Vol. 82, No. 5, December 2009

MATHEMATICS MAGAZINE

Modeling a Dive into the Monty Hall Problem

- Casting Light on Cube Dissections
- The Monty Hall Problem, Reconsidered
- Modeling a Diving Board

An Official Publication of The MATHEMATICAL ASSOCIATION OF AMERICA

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Cover image: Modeling a Dive into the Monty Hall Problem, by Hunter Cowdery, supervised by Jason Challas. Hunter is diving into illustration at San Jose State University, after studying art with Jason at West Valley College.

The editor wishes to thank Jason Challas for providing clever, eye-catching images, created by him or his students, for *thirty* issues of the MAGAZINE.

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Vol. 82, No. 5, December 2009

MATHEMATICS MAGAZINE

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MATHEMATICS MAGAZINE (ISSN 0025-570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, D.C. 20036 and Montpelier, VT, bimonthly except July/August.
The annual subscription price for *MATHEMATICS MAGAZINE* to an individual member of the Association is $131. Student and unemployed members receive a 66% dues discount; emeritus members receive a 50% discount; and new members receive a 20% dues discount for the first two years of membership.)

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Postmaster: Send address changes to Membership/ Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036-1385.

Printed in the United States of America

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Casting Light on Cube Dissections

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doi:10.4169/002557009X478346

Dissections of squares that illustrate the Pythagorean Theorem are a bright spot in the field of geometric dissections. In particular, there are two infinite classes of Pythagorean triples for which we have uniform methods for finding 4-piece dissections of squares, and four pieces is the fewest that can be expected under a very reasonable assumption. Yet when we step up from dissecting squares to dissecting cubes, prospects seem to dim. No infinite classes have been identified that involve an arguably minimum number of pieces. Only a handful of individually impressive examples have been found.

In this article we bring cube dissections out of the shadows by identifying the first such infinite class. We connect our class with hundred-year-old observations of French and English mathematicians, with a recently identified family of dissections of squares, and with a single previously known and arguably minimal dissection of cubes. In the process, we highlight some beautiful symmetry and provide new impetus to search for other classes of cube identities that could also yield minimal dissections.

Just how beautiful is the symmetry? Let's sneak a peak in FIGURE 1 at Michael Boardman's dissection of squares [**4**], which demonstrates the identity $3^2 + 4^2 = 5^2$. Boardman sliced the 4-square into four congruent rectangles that fit against the four sides of the 3-square to produce the 5-square. It would seem to be difficult to beat such natural symmetry!

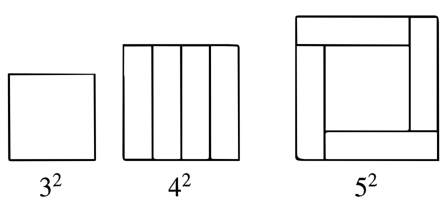

Figure 1 Boardman's dissection of squares for $3^2 + 4^2 = 5^2$

When we step up to three dimensions, we will first train our spotlight on the identity $1^3 + 1^3 + 5^3 + 6^3 = 7^3$. Analogous to FIGURE 1, FIGURE 2 has the 6-cube cut into six congruent rectangular blocks that cover the six surfaces of the 5-cube. Adding the two 1-cubes at opposite corners completes the 7-cube. Exploded versions of the 6- and 7-cube in FIGURE 3 help show how the pieces come together.

We will first survey relevant integer identities and the corresponding square and cube dissections, and next review the derivation of an infinite class of square identities

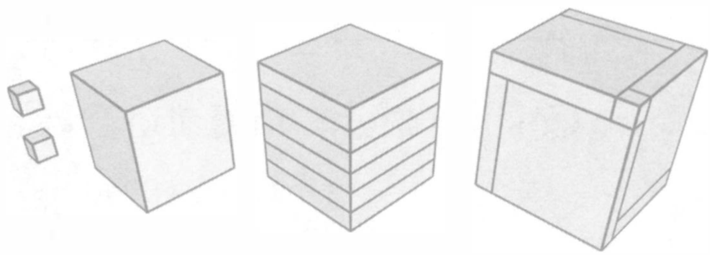

Figure 2 Symmetrical dissection of cubes for $1^3 + 1^3 + 5^3 + 6^3 = 7^3$

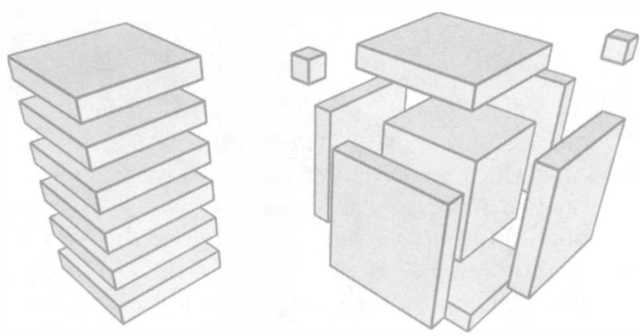

Figure 3 Exploded 6- and 7-cubes for symmetrical $1^3 + 1^3 + 5^3 + 6^3 = 7^3$

and their associated dissections. We then derive the cube identities and find lovely dissections for them. Finally, we will identify related families of cube identities and dissections for them as well.

Perhaps the most famous identity is $x^2 + y^2 = z^2$, which represents the Pythagorean Theorem. Integral solutions are called *Pythagorean triples*, with the most recognizable, $3^2 + 4^2 = 5^2$, probably known to the ancient Babylonians and Egyptians. Proclus attributed the class of solutions with $z = y + 1$ to Pythagoras and the class with $z = y + 2$ to Plato [**1**]. Diophantus found a method to generate all basic solutions [**18**]. Leonard E. Dickson provided an exhaustive survey of sums of squares that equal other sums of squares [**9**].

The earliest mention of a specific sum of cubes equaling another cube is of $3^3 + 4^3 + 5^3 = 6^3$, by Pietro Bongo [**5**] and François Viète [**24**] in 1591. Viète gave a formula that generated many but not all solutions to $x^3 + y^3 + z^3 = w^3$. Again, Dickson provided an exhaustive summary of results on sums of cubes that equal other sums of cubes.

Once such identities were in hand, it would be only a matter of time until people started searching for dissections of either squares or cubes to demonstrate the identities. Around the turn of the 20th century, the American puzzlist Sam Loyd and the English puzzlist Henry Dudeney published 4-piece dissections of squares for $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$ in their mathematical puzzle columns [**11, 19**] and in their books [**12, 13, 20**]. In fact, general techniques in [**14**] give us 4-piece dissections of squares for two infinite families of Pythagorean triples, namely the two previously-mentioned classes of Pythagoras and Plato.

As for dissecting cubes, Loyd and Dudeney did not attempt even the simplest such identity, $3^3 + 4^3 + 5^3 = 6^3$. It would be a half century until Herbert W. Richmond, an algebraic geometer at Cambridge University, published a 12-piece dissection [**21**]. A decade later, John Leech and Roger Wheeler, two math students at Cambridge, respectively asked for a dissection of at most ten pieces [**2**], and found an 8-piece dissection [**3**]. Many such solutions, some quite elaborate, have since been found by Edo Timmermans [**15, 22, 23**], using sophisticated computer search methods. As for other cube identities, ingenious dissections of cubes for $1^3 + 6^3 + 8^3 = 9^3$, $12^3 + 1^3 = 10^3 + 9^3$, and two others have collectively been given by James H. Cadwell, Robert Reid, and Gavin Theobald [**6, 14, 16**]. In no case do the techniques suggest infinite families.

Dostor's identities and Boardman's dissections

Our cube dissections are inspired by a family of dissections of squares that Michael Boardman [**4**] used to demonstrate identities in the sequence

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$\vdots$$

The first to study these identities was Georges Dostor [**10**], a 19th-century French mathematician. He found a simple way to produce them: Since the nth identity has $2n + 1$ terms, he chose x^2 as the middle term, so that the identity was then

$$(x-n)^2 + \cdots + (x-1)^2 + x^2 = (x+1)^2 + \cdots + (x+n)^2.$$

Isolating the x^2 term yielded

$$\begin{aligned} x^2 &= \left((x+1)^2 - (x-1)^2\right) + \cdots + \left((x+n)^2 - (x-n)^2\right) \\ &= 4x\sum_{i=1}^{n} i = 4x(n+1)n/2 = 2x(n+1)n. \end{aligned}$$

Thus $x = 2n(n+1)$.

Boardman created his dissections by cutting the x-square into $4n$ rectangles, with four rectangles of dimension $i \times x$ for each $i = 1, 2, \ldots n$. He produced the $(x+i)$-square by arranging the four $(i \times x)$-rectangles around the $(x-i)$-square.

We have already seen, in FIGURE 1, Boardman's dissection for the first identity, with $n = 1$ and $x = 4$. FIGURE 4 displays the crucial 12-, 13-, and 14-squares in Boardman's dissection for the second identity, for which $n = 2$ and $x = 12$. With the 10- and 11-squares remaining uncut, this visual argument readily establishes that

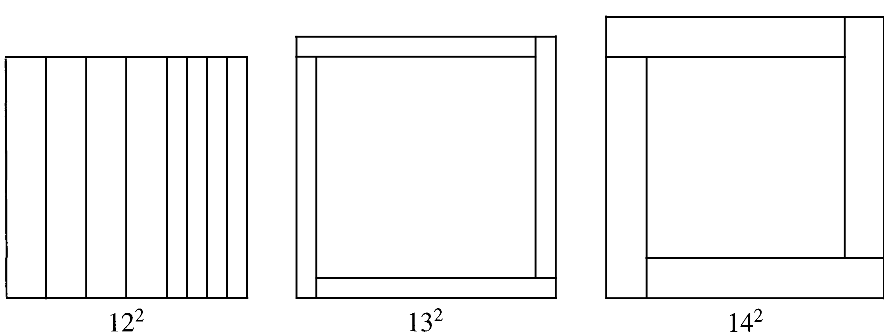

Figure 4 12-, 13-, and 14-squares for Boardman's $10^2 + 11^2 + 12^2 = 13^2 + 14^2$

$10^2 + 11^2 + 12^2 = 13^2 + 14^2$. Analogously, any other identity in the sequence will be confirmed by its corresponding dissection. For the nth identity, the number of pieces is $5n$.

As beautiful as they are, these dissections do not have the fewest possible pieces. In *Math Horizons* [**17**], I described alternative dissections that require only $4n$ pieces. I proved that $4n$ is the fewest possible whenever the cuts must consist of straight line segments, each of which is restricted to be parallel to some side of the square. Why bother to minimize the number of pieces? Otherwise, a dissection of squares for $3^2 + 4^2 = 5^2$ could consist of 25 unit squares—not particularly elegant! To achieve the minimum, we may need to trade away any hope of symmetry, as I did in [**17**]. Or we may try to minimize the number of pieces, subject to preserving symmetry, as in FIGURE 2.

A new class of cube identities

Disappointingly, none of the cube identities $x^3 + y^3 + z^3 = w^3$ seem to lead to a class of cube identities as cooperative as Dostor's square identities. It was left to the seemingly odd identity $1^3 + 1^3 + 5^3 + 6^3 = 7^3$, for which Robert Reid had found an 8-piece dissection [**14**], to illuminate the way.

Ignoring the two 1-cubes for the moment, we see the consecutive positive integers 5, 6, and 7. Let's try to apply Dostor's approach to sums of cubes and try to solve

$$(x - n)^3 + \cdots + (x - 1)^3 + x^3 = (x + 1)^3 + \cdots + (x + n)^3.$$

The 19th-century French mathematician Edouard Collignon proved that there is no positive integral solution for this equation [**8**]. To dodge the gloom, let's change the "=" to a "vs." and expand the above to give

$$\begin{array}{ccc} x^3 - 3nx^2 + 3n^2x - n^3 + \cdots & & x^3 + 3x^2 + 3x + 1 + \cdots \\ & \text{vs.} & \\ + x^3 - 3x^2 + 3x - 1 + x^3 & & + x^3 + 3nx^2 + 3n^2x + n^3. \end{array}$$

After canceling, we note that there are extra terms that do not contain x as a factor, namely, $-n^3 - (n - 1)^3 - \cdots - 1$ on the left and $n^3 + (n - 1)^3 + \cdots + 1$ on the right. Since these extra terms are cubes, let's include them, so that we arrive at

$$\begin{aligned} 1^3 + 1^3 + \cdots + n^3 + n^3 + (x - n)^3 + \cdots + (x - 1)^3 + x^3 \\ = (x + 1)^3 + \cdots + (x + n)^3. \end{aligned}$$

With this general form, isolating the x^3 term yields

$$x^3 = 6x^2 \sum_{i=1}^{n} i = 6x^2(n + 1)n/2 = 3x^2(n + 1)n.$$

Thus $x = 3n(n + 1)$.

We then get the following sequence of identities, noting that for $n = 1$, we have precisely the identity for Reid's dissection:

$$1^3 + 1^3 + 5^3 + 6^3 = 7^3$$
$$1^3 + 1^3 + 2^3 + 2^3 + 16^3 + 17^3 + 18^3 = 19^3 + 20^3$$
$$1^3 + 1^3 + 2^3 + 2^3 + 3^3 + 3^3 + 33^3 + 34^3 + 35^3 + 36^3 = 37^3 + 38^3 + 39^3$$
$$\vdots$$

Dissections for the cube identities

We can create dissections for these identities by cutting the x-cube into $6n$ rectangular blocks, with six rectangular blocks of dimension $i \times x \times x$ for each $i = 1, 2, \ldots, n$. We then produce the $(x+i)$-cube by placing the $(x-i)$-cube in the middle, arranging the six $(i \times x \times x)$-rectangular blocks around it, and finally adding in the two i-cubes at opposite corners.

In FIGURES 2 and 3 we have already seen the dissection for the first identity, with $n = 1$ and $x = 6$. For $n = 2$, and thus $x = 18$, we see in FIGURE 5 the assembled 19- and 20-cubes from a symmetrical dissection for the second identity. Note that the pieces are not to the same scale as the pieces in FIGURES 2 and 3. The basic structure for each of these two cubes is essentially the same as for the 7-cube in FIGURE 2. Exploded views of the 19- and 20-cubes hover in FIGURE 6. This visual argument makes it easy to see that $1^3 + 1^3 + 2^3 + 2^3 + 16^3 + 17^3 + 18^3 = 19^3 + 20^3$. The attractive symmetry so evident in FIGURES 5 and 6 echoes that of Boardman's dissection in FIGURE 4. We can expect the same for any other identity in the sequence. For the nth identity, the number of pieces is $9n$.

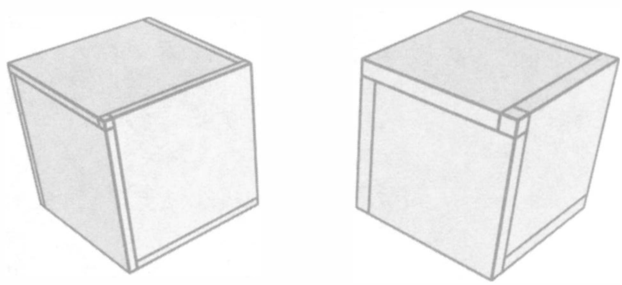

Figure 5 The 19- and 20-cubes from a symmetrical dissection of cubes for $1^3 + 1^3 + 2^3 + 2^3 + 16^3 + 17^3 + 18^3 = 19^3 + 20^3$

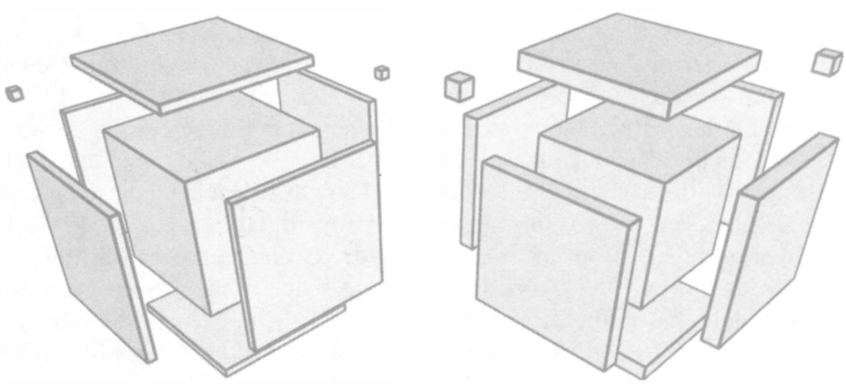

Figure 6 Exploded 19- and 20-cubes for $1^3 + 1^3 + 2^3 + 2^3 + 16^3 + 17^3 + 18^3 = 19^3 + 20^3$

Apparently without being aware of either the infinite class of cube identities or the 9-piece symmetric dissection in FIGURES 2 and 3, Robert Reid found the clever 8-piece dissection of cubes for $1^3 + 1^3 + 5^3 + 6^3 = 7^3$ shown in FIGURES 7 and 8. He placed the 5-cube in one corner of the 7-cube, and cut the 6-cube into four rectangular blocks, of dimensions $(2 \times 6 \times 6)$, $(2 \times 5 \times 6)$, $(2 \times 1 \times 6)$, and $(2 \times 6 \times 6)$. He next arranged the first three blocks to fill in the remaining three corners for one side of the 7-cube. He then cut from the remaining $(2 \times 6 \times 6)$ block an L-shaped piece with each

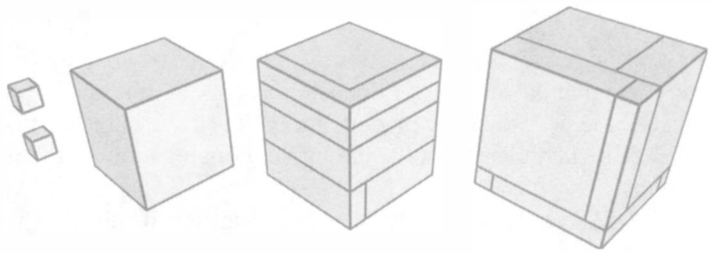

Figure 7 Reid's dissection of cubes for $1^3 + 1^3 + 5^3 + 6^3 = 7^3$

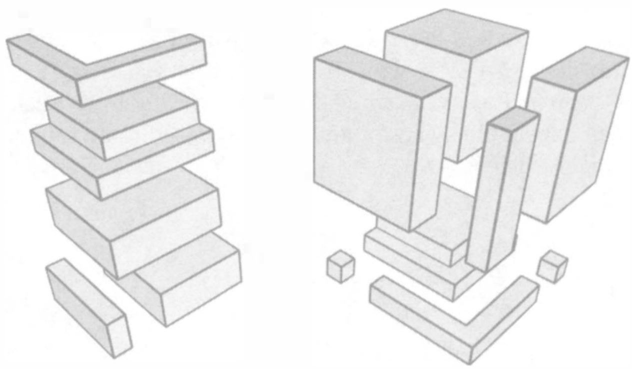

Figure 8 Exploded 6- and 7-cubes for Reid's $1^3 + 1^3 + 5^3 + 6^3 = 7^3$

arm of thickness 1 and length 6, leaving a piece in the shape of a stepped platform. The platform-shaped piece fits tightly against the 5-cube and neighboring blocks from the 6-cube. Finally, the L-shaped piece, with a 1-cube sitting at the end of each arm, fills the remaining space in the 7-cube.

For any $n > 1$, we generalize Reid's dissection method to create dissections for the nth identity in the following way. For each $i = 1, 2, \ldots, n$, place the $(x - i)$-cube in one corner of the $(x + i)$-cube, and cut out of the x-cube four rectangular blocks, of dimensions $(2i \times x \times x)$, $(2i \times x \times x)$, $(2i \times (x - i) \times x)$, and $(2i \times i \times x)$. Arrange the latter three blocks to fill in the remaining three corners for one side of the $(x + i)$-cube. Then take the remaining $(2i \times x \times x)$ block, and cut from it an L-shaped piece with each arm of thickness i and length x, leaving a piece in the shape of a stepped platform. The platform-shaped piece fits tightly against the $(x - i)$-cube and neighboring blocks from the x-cube, and the L-shaped piece, with the i-cubes sitting at the end of each arm, fills the remaining space in the $(x + i)$-cube. We see the corresponding dissection for $n = 2$, namely for $1^3 + 1^3 + 2^3 + 2^3 + 16^3 + 17^3 + 18^3 = 19^3 + 20^3$ in FIGURES 9 and 10. The total number of pieces is 8 for each of the cubes larger than the x-cube, and thus $8n$ pieces altogether.

Thus by sacrificing some symmetry, as in FIGURES 7 and 8 and in FIGURES 9 and 10, we are able to reduce the number of pieces from $9n$ to $8n$. We shall see in the next section that this is the best that we can hope for, under a very reasonable assumption.

Minimality and more families

Summarizing, for any $n \geq 1$, we have a dissection of cubes for the nth identity that uses $8n$ pieces. Is it possible to further reduce the number of pieces? No, at least not if we consider dissections in which every cut in a cube is a polygonal surface such

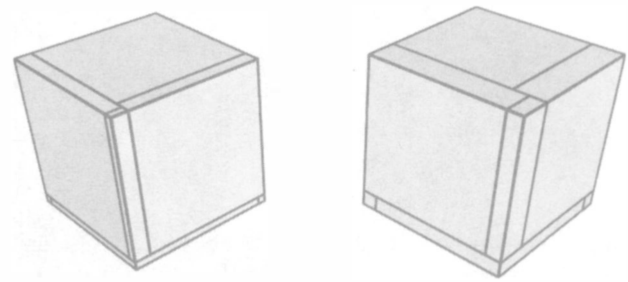

Figure 9 The 19- and 20-cubes from a Reid-like dissection of cubes for $1^3 + 1^3 + 2^3 + 2^3 + 16^3 + 17^3 + 18^3 = 19^3 + 20^3$

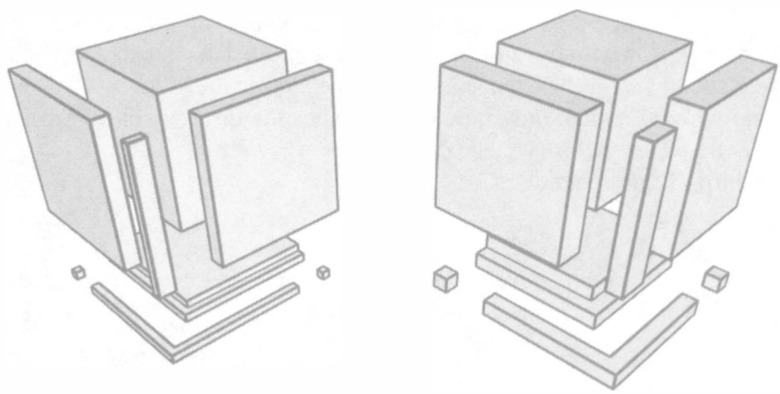

Figure 10 Exploded 19- and 20-cubes from a Reid-like dissection of cubes for $1^3 + 1^3 + 2^3 + 2^3 + 16^3 + 17^3 + 18^3 = 19^3 + 20^3$

that each polygon is parallel to some face of that cube. The reason is that any z-cube larger than the x-cube will be formed by pieces whose extent in a direction parallel to an edge of the cube is at most x. It follows that no two corners of the z-cube can be filled simultaneously by the same piece. With at least eight pieces for each of the n cubes that are larger than the x-cube, such a dissection must have at least $8n$ pieces. This same sort of argument was given in [**14**, pp. 249–250].

Are we justified in restricting every cut in a cube to be a surface consisting of polygons each of which is parallel to some face of that cube? Relaxing that restriction does not seem to help, at least for dissections of three cubes to one, because in this case, there are no known 7-piece dissections, and all known 8-piece dissections have cuts consisting of polygons that are parallel to the faces of the cubes.

Are there other exotic cube identities aside from those that we have presented? How about a cube identity in which there are $2n + 2$ positive integers aside from those for the small cubes? Let's make the two integers in the middle be identical and the rest of the integers be consecutive. Can we find such identities for which the sum of the $2n$ small filler cubes, plus the next $n + 2$ cubes, equals the sum of the remaining n cubes? Once again applying a modified Dostor's approach, we get the identity

$$1^3 + 1^3 + \cdots + n^3 + n^3 + (x - n)^3 + \cdots + (x - 1)^3 + x^3 + x^3$$
$$= (x + 1)^3 + \cdots + (x + n)^3$$

from which we can conclude that $x = 3n(n + 1)/2$.

We then get the following sequence of identities, noting that for $n = 1$, we have lifted from obscurity an identity published a century ago by Robert W. D. Christie, at

Stanley College in Liverpool, England [7]:

$$1^3+1^3+2^3+3^3+3^3=4^3$$
$$1^3+1^3+2^3+2^3+7^3+8^3+9^3+9^3=10^3+11^3$$
$$1^3+1^3+2^3+2^3+3^3+3^3+15^3+16^3+17^3+18^3+18^3=19^3+20^3+21^3$$

There are symmetrical $9n$-piece dissections of cubes for these identities, too. When $n=1$, just slice each of the two 3-cubes into three $1\times 3\times 3$ rectangular blocks, place each of those six blocks against a face of the 2-cube, and add the two 1-cubes at opposite corners of the resulting 4-cube.

Because there are two x-cubes rather than one, it may not always be possible to hit the lower bound of $8n$ pieces for all values of n. Whenever x is even, that is, when $\lceil n/2\rceil$ is even, we can achieve an $(8n)$-piece dissection. This is because the Reid-like dissection then allows for the even split of volume among the $(2i\times x\times x)$-rectangular blocks. And when x is odd, one more cut produces the desired split between the two x-cubes, giving us an $(8n+1)$-piece dissection.

For the third of these identities,

$$1^3+1^3+2^3+2^3+3^3+3^3+15^3+16^3+17^3+18^3+18^3=19^3+20^3+21^3,$$

$\lceil n/2\rceil=\lceil 3/2\rceil=2$, which is even. However, we don't need to display the dissection, because the above identity is the sum of the identity

$$1^3+1^3+2^3+2^3+16^3+17^3+18^3=19^3+20^3$$

for which we gave a 16-piece dissection in FIGURE 9, plus 3^3 times the identity

$$1^3+1^3+5^3+6^3=7^3$$

for which we displayed an 8-piece dissection in FIGURE 7. Combining the 8-piece and 16-piece dissections gives the desired 24-piece dissection!

And for $n=1$, even though $\lceil n/2\rceil$ is odd, we can find an $8n$-piece dissection by deviating a bit from the standard Reid-like dissection. Cut a $(1\times 2\times 3)$-rectangular block out of one 3-cube. Cut the other 3-cube into a $(1\times 3\times 3)$-rectangular block and an L-shaped piece with each arm of thickness 1 and length 3, leaving a piece in the shape of a stepped platform. We see the resulting 8-piece dissection in FIGURE 11 and an exploded version in FIGURE 12.

There are still many possibilities for infinite families of cube identities. Suppose we have three copies of x^3, rather than two or one. Then we discover that $x=(n+1)n$. For $n=1$, this gives the identity

$$1^3+1^3+1^3+2^3+2^3+2^3=3^3.$$

It is easy to find a corresponding symmetric 9-piece dissection and not so hard to find a corresponding 8-piece dissection. Moreover, there is no rule that requires the large numbers in an identity to be in arithmetic progression. Thus we can base a class of identities on:

$$\cdots+(x-6)^3+(x-3)^3+(x-1)^3+x^3 \text{ vs. } (x+1)^3+(x+3)^3+(x+6)^3+\cdots.$$

Our techniques work well for the corresponding dissections, as you can easily verify.

Animations that illuminate some of the dissections in this article are posted at the MAGAZINE website and at `http://www.cs.purdue.edu/homes.gnf/book/cubes.html`. We hope this will be the dawn of a bright new day for cube dissections. Just be sure to wear your sunglasses!

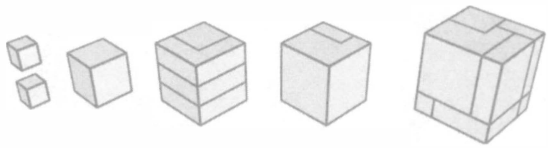

Figure 11 Semi-Reid-like dissection of cubes for $1^3 + 1^3 + 2^3 + 3^3 + 3^3 = 4^3$

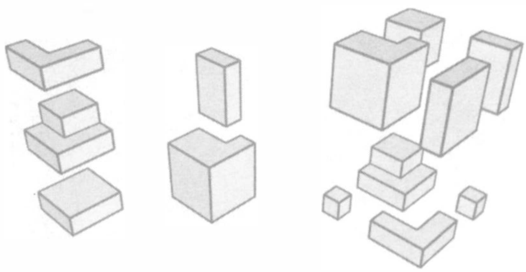

Figure 12 Exploded semi-Reid-like dissection of cubes for $1^3 + 1^3 + 2^3 + 3^3 + 3^3 = 4^3$

Acknowledgment. I am pleased to thank Isabel McMann, at the Radcliffe Science Library, University of Oxford, for supplying a copy of the article by Robert Christie. Thanks to Gavin Theobald for catching some typos. Finally, I would like to thank the referees for helpful suggestions.

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The Monty Hall Problem, Reconsidered

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doi:10.4169/002557009X478355

In its classical form, the Monty Hall Problem (MHP) is the following:

VERSION 1. (CLASSIC MONTY) You are a player on a game show and are shown three identical doors. Behind one is a car, behind the other two are goats. Monty Hall, the host of the show, asks you to choose one of the doors. You do so, but you do not open your chosen door. Monty, who knows where the car is, now opens one of the doors. He chooses his door in accordance with the following rules:

1. Monty always opens a door that conceals a goat.
2. Monty never opens the door you initially chose.
3. If Monty can open more than one door without violating rules one and two, then he chooses his door randomly.

After Monty opens his door, he gives you the choice of sticking with your original choice or switching to the other unopened door. What should you do to maximize your chances of winning the car?

In the entire annals of mathematics, you would be hard-pressed to find a problem that arouses the passions like the MHP. It has a history going back at least to 1959, when Martin Gardner introduced a version of it in *Scientific American* [**4, 5**]. When statistician Fred Moseteller included it in his 1965 anthology of probability problems [**9**], he remarked that it attracted far more mail than any other problem. In his 1968 book *Mathematical Ideas in Biology* [**18**], biologist John Maynard Smith wrote, "This should be called the Serbelloni problem since it nearly wrecked a conference on theoretical biology at the villa Serbelloni in the summer of 1966." In its modern game show format the problem made its first appearance in a 1975 issue of the academic journal *The American Statistician* [**16**]. Mathematician Steve Selvin presented it as an interesting classroom exercise on conditional probability. Though he presented the correct solution, (that there is a big advantage to be gained from switching), he found himself strongly challenged by subsequent letters to the editor [**17**].

The problem really came into its own when *Parade* magazine columnist Marilyn vos Savant responded to a reader's question regarding it. There followed several rounds of angry correspondence, in which readers challenged vos Savant's solution. The challengers later had to eat crow when it was shown by a Monte Carlo simulation that vos Savant was correct, but not before the fracas reached the front page of the *New York*

Times [**20**]. The whole story is recounted in the books by Rosenhouse and vos Savant [**13, 15**].

In the end, the situation has been best summed up by cognitive scientist Massimo Palmatelli-Palmarini who wrote that, "... no other statistical puzzle comes so close to fooling all the people all the time... The phenomenon is particularly interesting precisely because of its specificity, its reproducibility, and its immunity to higher education" [**10**].

Why all the confusion?

The trouble, you see, is that most people argue like this: "Once Monty opens his door only two doors remain in play. Since these doors are equally likely to be correct, it does not matter whether you switch or stick." We will refer to this as the fifty-fifty argument.

This intuition is supported by a well-known human proclivity. A negative consequence incurred by inaction hurts less than the same negative consequence incurred through some definite action. In the context of the MHP, people feel worse when they switch and lose than they do after losing by sticking passively with their initial choice.

There is a large literature in the psychology and cognitive science journals documenting and explaining the difficulty people have with the MHP. Burns and Wieth [**3**] summarized the findings of numerous such studies by writing,

> These previous articles reported 13 studies using standard versions of the MHD, and switch rates ranged from 9% to 23% with a mean of 14.5%. This consistency is remarkable given that these studies range across large differences in the wording of the problem, different methods of presentation, and different languages and cultures.

(Note that MHD stands for "Monty Hall Dilemma.")

Gilovich, Medvec, and Chen [**6**] studied people's reactions to losing by switching versus their reactions to losing by sticking. They used boxes instead of doors, and crafted an experimental situation in which players would lose regardless of their decision to switch or stick. Their findings?

> Because action tends to depart from the norm more than inaction, the individual is likely to feel more personally responsible for an unfortunate action. Thus, subjects who switched boxes in our experiment were more likely to experience a sense of "I brought this on myself," or "This need not have happened," than subjects who decided to keep their initial box.

It would seem the defenders of sticking can point both to a plausible mathematical argument and to certain fine points of human psychology. How can the forces for switching fight back?

Focus on Monty, not the doors There are a variety of elementary methods for solving the MHP. Working out a tree diagram for the problem, as in FIGURE 1, establishes that switching wins with probability $\frac{2}{3}$, while sticking wins with probability $\frac{1}{3}$. Consequently, we double our chances of winning by switching.

Monte Carlo simulations are also effective for establishing the correct answer. The Monty Hall scenario is readily simulated on a computer. The large advantage to be gained from switching quickly becomes apparent by playing the game multiple times.

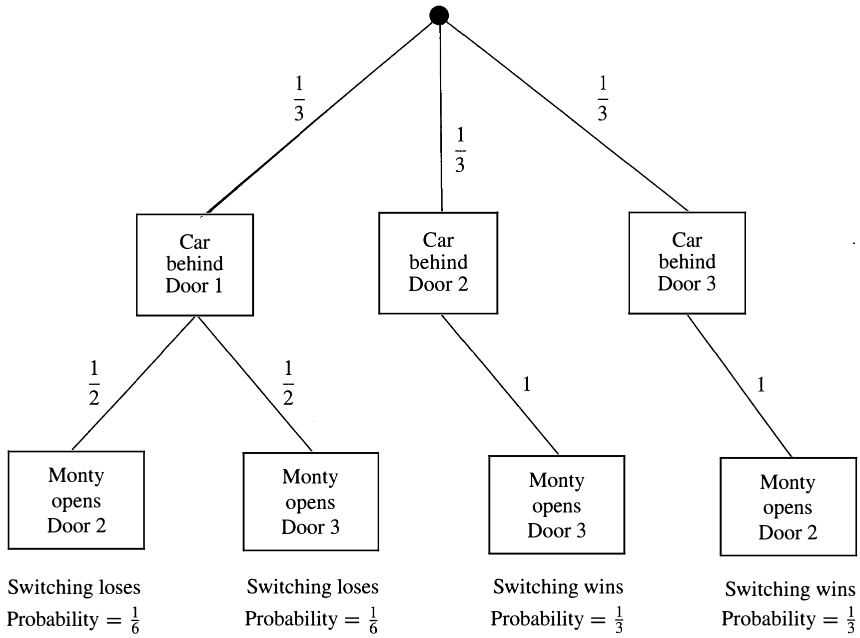

Figure 1 Probability tree for the classical MHP when the player initially chooses door one

Such methods, however, do little to clarify why the fifty-fifty argument is incorrect. Practical results obtained from a simulation can show you that *something* is wrong with your intuition, but they will not make the correct answer seem natural. The trouble lies in the difficulty people have in recognizing what is and is not important in reasoning about conditional probability.

The mantra about focus goes a long way towards pointing people in the right direction. When Monty opens door X, there is a tendency to think, "I have learned that door X conceals a goat, but I have learned nothing of relevance about the other two doors." This is what we mean by "focusing on the doors." The proper approach involves focusing on Monty, specifically on the precise manner in which he chooses his door to open. We should think, "Monty, who makes his decisions according to strict rules, chose to open door X. Why this door as opposed to one of the others?"

Let us assume the player initially chose door one and Monty then opened door two. According to the rules, we can be certain that one of the following two scenarios has played out:

1. The car is behind door one. Monty chose door two at random from among doors two and three.
2. The car is behind door three. Since the player initially chose door one, Monty was now forced to open door two.

The second of these scenarios is more likely than the first. Since the car is behind the first door one-third of the time, and since Monty then opens door two in one-half of those cases, we see that scenario one occurs one-sixth of the time. Scenario two, on the other hand, happens whenever the car is behind door three (and the player has chosen door one). That happens one-third of the time. Scenario two is twice as likely as scenario one.

Thus, we should think, "I have just witnessed an event that is twice as likely to occur when the car is behind door three than it is when the car is behind door one.

Consequently, the car is more likely to be behind door three, and I am more likely to win the car by switching."

An exotic selection procedure The general principle here is that anything affecting Monty's decision-making process is relevant to updating our probabilities after Monty opens his door. To further illuminate this point, let us consider an altered version of the problem:

VERSION 2. (HIGH-NUMBERED MONTY) As before, we have three identical doors concealing one car and two goats. The player chooses a door that remains unopened. Monty now opens a door he knows to conceal a goat. This time, however, we stipulate that Monty always opens the highest-numbered door available to him (keeping in mind that Monty will never open the door the player chose.) Will the player gain any advantage by switching doors?

For reasons of concreteness, we will assume once more that the player initially chooses door one.

Any time door one conceals a goat, Monty has no choice regarding which door to open. He can not open door one (since the player chose that door), and he can not open the door that conceals the car. This leaves only one door available to him.

The interesting cases occur when door one conceals the car. Unlike Classic Monty, who now chooses randomly, High-Numbered Monty will always open door three when he can. It follows that if we see him open door two instead we know for certain that the car is behind door three.

And if High-Numbered Monty opens door three? Since Monty is certain to open door three whenever the car is behind door one or door two, we now have no basis for deciding between them. It really is a fifty-fifty decision in this case.

Take this as a cautionary tale. Whether we are playing Classic Monty or High-Numbered Monty, it is certain that Monty will open a goat-concealing door. In the former case the probability that our initial choice concealed the car did not change while in the latter case it did. This shows that any proposed solution to the MHP failing to pay close attention to Monty's selection procedure is incomplete.

Monty meets Bayes

The main point thus far is that the probability that door X conceals the car, given that Monty has shown us the goat behind door Y, depends on a detailed consideration of Monty's selection procedure. More precisely, it depends on the probability that Monty will open door Y under the assumption that door X conceals the car. The precise manner in which these probabilities are related is given by Bayes' theorem.

We denote by C_i the event that the car is behind door i, and by M_j the event that Monty opens door j to reveal a goat. Also assume the player initially chooses door one, and Monty then opens door two. The probability that the player's door conceals the car, given Monty's display, can be found using Bayes' theorem:

$$P(C_1|M_2) = \frac{P(C_1)P(M_2|C_1)}{P(M_2)}.$$

Expanding the bottom of this fraction via the law of total probability leads to

$$P(C_1|M_2) = \frac{P(C_1)P(M_2|C_1)}{P(C_1)P(M_2|C_1) + P(C_2)P(M_2|C_2) + P(C_3)P(M_2|C_3)}.$$

In both of our versions of the MHP we have $P(M_2|C_2) = 0$, since it is given that Monty will never open the door concealing the car. Also, since we are given that the doors are identical, we have

$$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}.$$

Making these substitutions leads to

$$P(C_1|M_2) = \frac{P(M_2|C_1)}{P(M_2|C_1) + P(M_2|C_3)}.$$

In both versions of the game we have $P(M_2|C_3) = 1$. That is, when the player chooses door one and the car is behind door three, Monty is certain to open door two.

In Classic Monty we have $P(M_2|C_1) = \frac{1}{2}$, since Monty chooses at random when the car is behind the door initially chosen by the player. In High-Numbered Monty we have $P(M_2|C_1) = 0$, since Monty is required by his rules to open door three. Plugging everything into Bayes' Theorem shows that for Classic Monty we now have

$$P(C_1|M_2) = \frac{\frac{1}{2}}{\frac{1}{2}+1} = \frac{1}{3} \quad \text{[Classic]},$$

while for High-Numbered Monty we have

$$P(C_1|M_2) = \frac{\frac{1}{2}}{0+1} = \frac{1}{2} \quad \text{[High-Numbered]}.$$

These are precisely the answers we obtained in the previous section.

Let us go one more round:

VERSION 3. (RANDOM MONTY) As always, assume that the player has initially chosen door one and Monty subsequently opened door two to reveal a goat. This time, however, you know that Monty chose his door randomly, subject only to the restriction that he not open the door the player chose. Should we switch?

The novelty here lies in the nonzero probability of Monty opening the door concealing the car. Intuitively we would reason as follows: Since Monty opened door two after I selected door one, since door two concealed a goat, and since I know Monty chose randomly between doors two and three, I conclude that one of two scenarios has played out:

1. The car is behind door one, Monty chose door two randomly.
2. The car is behind door three, Monty chose door two randomly.

Since the car is equally likely to be behind doors one and three, these scenarios are equally likely to occur. The conclusion is that the remaining doors are equiprobable, and therefore there is no advantage to switching.

Our intuition is confirmed via Bayes' Theorem. We know that Monty will not open door one, and we know that door two conceals a goat. We now have

$$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3},$$

$$P(M_2|C_1) = P(M_2|C_3) = \frac{1}{2}$$

$$P(M_2|C_2) = 0.$$

Bayes' Theorem now says

$$P(C_1|M_2) = \frac{\frac{1}{3}\left(\frac{1}{2}\right)}{\frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}(0) + \frac{1}{3}\left(\frac{1}{2}\right)} = \frac{1}{2}.$$

The tree diagram in FIGURE 2 might be helpful for visualizing the situation.

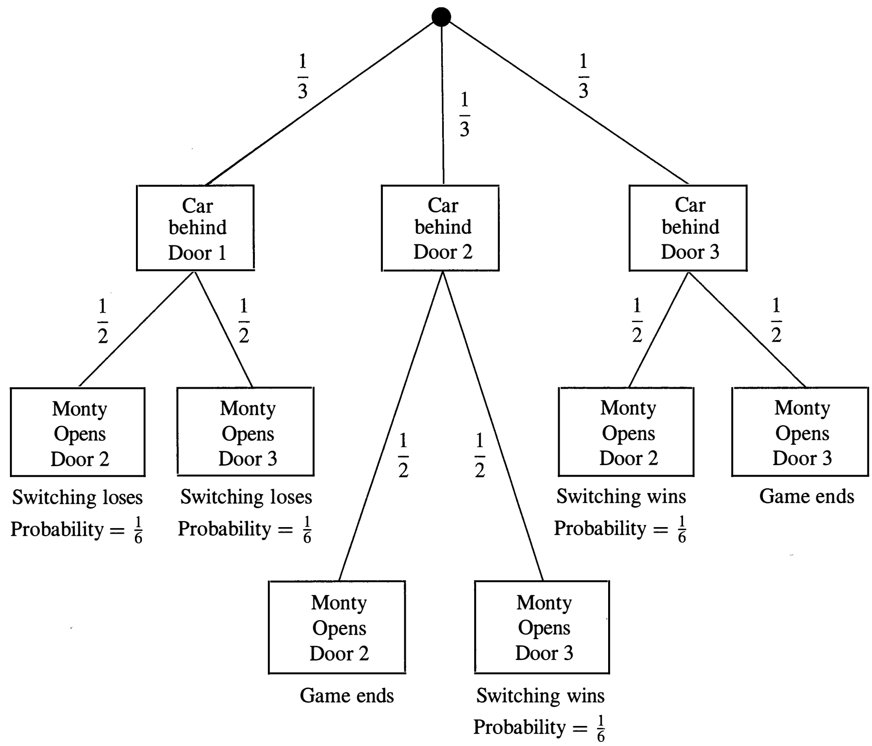

Figure 2 Probability tree for Random Monty when the player initially chooses door one

Two-player Monty

Three-door versions of the MHP can become remarkably complex. The following version comes from a paper by philosopher Peter Baumann [**1, 2**]. For the remainder of the paper we will refer simply to the probability of door X, thereby avoiding the more cumbersome expression, "The probability that door X conceals the car."

VERSION 4. (TWO-PLAYER MONTY) We begin with three identical doors concealing two goats and one car. There are two players in the game. Each player chooses one of the doors, but does not open it. Each player knows there is another person in the game, but neither knows which door the other player selected. Monty now opens a door according to the following procedure.

1. If both players selected the same door, then everything proceeds as in the classical game. Monty opens a goat-concealing door, choosing randomly if he has a choice.
2. If the players selected different doors, then Monty opens the one remaining door, regardless of what is behind it.

We assume that both players select their initial doors randomly. If you are one of the players and you have just seen Monty open a goat-concealing door, should you switch?

This will be a fine test of our new-found intuition. For concreteness, suppose that Player A initially chose door one, and Monty has now opened the goat-concealing door two. What do Monty's actions tell us about Player B's choice? Initially we consider it equally likely that Player B chose door one, door two or door three. After seeing Monty open door two we reason that one of three scenarios has played out:

1. Player B chose door three. In this case Monty was forced to open door two, which conceals a goat with probability $\frac{2}{3}$.
2. Player B chose door one and door one conceals the car. In this case Monty opens door two with probability $\frac{1}{2}$. Since door one conceals the car with probability $\frac{1}{3}$, this scenario occurs with probability $\frac{1}{6}$.
3. Player B chose door one and door three conceals the car (which happens with probability $\frac{1}{3}$). In this case Monty is again forced to open door two.

Combining items two and three above shows that Player B chose door one with probability $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$. Item one shows that Player B chose door three with probability $\frac{2}{3}$. We conclude that the event in which Monty opens the goat-concealing door two after Player A chooses door one is $\frac{4}{3}$ more likely to occur when Player B has chosen door three than when he has chosen door one.

It is a consequence of Bayes' theorem that the probabilities we now assign to "Player B chose door three," and "Player B chose door one," must preserve this 4 : 3 ratio. (A proof of this assertion can be found in the paper by Rosenthal [**14**].) Consequently, we assign probabilities of $\frac{4}{7}$ and $\frac{3}{7}$ respectively.

To continue the analysis, note that from Player A's perspective there are now four possibilities. Player B could have chosen door one or door three, and the car could be behind either of those doors. Let us denote these possibilities via ordered pairs of the form

$$\text{(Player } B\text{'s Door, Location of the Car).}$$

Thus, the four remaining possibilities are

$$(3,1), \quad (3,3), \quad (1,1), \quad (1,3).$$

Consider the first two pairs. If Player B chose door three, then Monty was forced to open door two. Consequently, we learn nothing regarding the probability of doors one and three. Since these two scenarios collectively have a probability of $\frac{4}{7}$, and since they are equally likely, we now assign the following probabilities:

$$P(3,1)=P(3,3)=\frac{2}{7}.$$

The remaining two pairs, however, are not equiprobable. Suppose that Player B chose door one, just as Player A did. If the car is behind door one, then Monty chose door two randomly, which happens with probability $\frac{1}{2}$. If the car is behind door three, then Monty was forced to choose door two. It follows that it is twice as likely that the car is behind door three than that it is behind door two. Since these scenarios have a collective probability of $\frac{3}{7}$, we assign the following probabilities:

$$P(1,1)=\frac{1}{7} \quad \text{and} \quad P(1,3)=\frac{2}{7}.$$

Of the four scenarios, the two in which Player A wins by switching are $(1, 3)$ and $(3, 3)$. Since both have probability $\frac{2}{7}$, this gives a total probability of winning by switching of $\frac{4}{7}$. That is our solution.

The really amusing part is that both players will go through this analysis, and both will decide to switch doors. In those scenarios in which the players chose different doors, this implies that someone is definitely making the wrong decision. Such are the cruelties of probability.

Two-player Monty has also been discussed by Baumann **[1]**, Levy **[7]**, Rosenhouse **[13]**, and Sprenger **[19]**.

Many doors

Ready for the final exam?

VERSION 5. (PROGRESSIVE MONTY) This time there are n identical doors, concealing one car and $n - 1$ goats. The player chooses a door, but does not open it. Monty now opens a goat-concealing door, choosing randomly from among his options. The player is now given the choice of sticking or switching. The player makes his choice, but again does not open his chosen door. Monty opens another goat-concealing door. The player is again given the opportunity to stick or switch. This continues until Monty has opened $n - 2$ doors. The player makes his final selection, and wins whatever is behind his door. What strategy will maximize his chances of winning the car?

To help us get our bearings, let us try a case study. Suppose we begin with five doors. At any stage of the game we represent the probabilities of the doors, based on all available knowledge, via an ordered 5-tuple, which we call the *probability vector*. As the game begins, we have probability vector

$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

As always, let us assume the player chooses door one and Monty now opens door two. Our past experience suggests that the probability of our door does not change, and this is confirmed by Bayes' theorem. In the following calculation, the notation $\overline{C_i}$ denotes the event where the car is not behind door i.

We now compute

$$\begin{aligned} P(C_1|M_2) &= \frac{P(C_1)P(M_2|C_1)}{P(C_1)P(M_2|C_1) + P(\overline{C_1} \text{ and } \overline{C_2})P(M_2|\overline{C_1} \text{ and } \overline{C_2})} \\ &= \frac{\frac{1}{5}\left(\frac{1}{4}\right)}{\frac{1}{5}\left(\frac{1}{4}\right) + \frac{3}{5}\left(\frac{1}{3}\right)} = \frac{1}{5}. \end{aligned}$$

Since the other doors are identical and since their probabilities must sum to $\frac{4}{5}$, we now have probability vector

$$\left(\frac{1}{5}, 0, \frac{4}{15}, \frac{4}{15}, \frac{4}{15}\right).$$

What if we now switch to door three and then see Monty open door five? Known probabilities are now

$$P(C_3) = P(C_4) = P(C_5) = \frac{4}{15}$$

$$P(M_5|C_1) = P(M_5|C_4) = \frac{1}{2}$$

$$P(C_1) = \frac{1}{5} \quad \text{and} \quad P(M_5|C_3) = \frac{1}{3}.$$

If we use the law of total probability to write

$$P(M_5) = P(C_1)P(M_5|C_1) + P(C_3)P(M_5|C_3) + P(C_4)P(M_5|C_4) = \frac{29}{90},$$

and plug the results into Bayes' Theorem, the result is the probability vector

$$\left(\frac{9}{29}, 0, \frac{8}{29}, \frac{12}{29}, 0\right).$$

The probabilities of all the remaining doors went up.

What if Monty had opened door one after we switched to door three? The reader can supply the details that lead to the vector

$$\left(0, 0, \frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right).$$

Notice that the probability of door three went down, from $\frac{4}{15}$ to $\frac{1}{4}$. Our chosen door actually seems less likely as the result of Monty's actions. A surprising result!

Things get messy indeed in this version. Plainly we need some guidelines to aid our intuition.

The first principle is simple. Any time Monty chooses not to open a door different from your present choice, the probability of that door goes up. In our case study, Monty opened door two after we chose door one. The event, "Monty does not open door three," is more likely to happen when the car is behind door three than when it is elsewhere. Consequently, we will revise upward our probability of door three.

The second principle is that if the doors different from your present choice are equiprobable, then the probability of your choice does not change when Monty opens a door. In our case study, after Monty opened door two, we reason that the event, "Monty does not open door one," has probability one regardless of the location of the car. Consequently, we learn nothing from the occurrence of that event. The calculation in our case study confirms this intuition.

Why, though, does it matter that the other doors are equiprobable? The answer is that Monty's failure to open a door is not the only source of information to which we have access. The probability of the event, "Monty opens door X," depends in part on the probability of the event, "Door X conceals the car." Specifically, the more likely a door is to conceal the car, the less likely Monty is to open that door. Once more returning to our case study, we switched to door three at a moment when doors three through five were equiprobable and collectively very likely to conceal the car. By opening door five, Monty eliminated one element of this collection. This revelation does nothing to shake our confidence that the car is more likely to be found among doors three through five than it is to be found among any collection of three doors that includes door one. Consequently, we will revise upward the probability of our chosen door.

Why did the probability of door three go down when Monty opened door one? This one is harder to explain, but our calculation suggests the proper way to think about it. If all four doors had been equiprobable at the moment we switched to door three, then we would simply be playing Classic Monty on four doors. In that case, our chosen door would retain its $\frac{1}{4}$ probability after Monty opens a door. The probability vector we computed for our present situation is identical to what we would have obtained were we playing four-door Classic Monty. The implication is that by eliminating door one, Monty has essentially erased the prior history of the game. We are now faced with three doors that were equiprobable at the moment we chose among them, and these doors were among an ensemble of four doors, just as in four-door Classic Monty. That door one had a different probability from the other doors does not distinguish our situation in a relevant way.

This observation leads to our final clue. If we select a door at a moment when precisely k doors remain, the probability of that door can never be smaller than $\frac{1}{k}$. Even if we have been careless in extracting the maximum amount of information from Monty's actions, we still know the door was chosen from among k possibilities.

As a test of our principles, let us go another round with our case study. We left off with the player having chosen door three and with probability vector,

$$\left(0,\ 0,\ \frac{1}{4},\ \frac{3}{8},\ \frac{3}{8}\right).$$

Imagine that we now switch to door four.

If Monty now opens door three then only doors four and five remain in play. We would reason that these two doors were equiprobable at the moment we switched to door four, but that door four was selected from among three possibilities. We are, in effect, playing Classic Monty, and we would expect our updated probability vector to be

$$\left(0,\ 0,\ 0,\ \frac{1}{3},\ \frac{2}{3}\right).$$

The calculation is

$$\begin{aligned} P(C_4|M_3) &= \frac{P(C_4)P(M_3|C_4)}{P(C_4)P(M_3|C_4) + P(C_5)P(M_3|C_5)} \\ &= \frac{\frac{3}{8}\left(\frac{1}{2}\right)}{\left(\frac{3}{8}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{8}\right)(1)} = \frac{1}{3}. \end{aligned}$$

And if Monty opens door five instead? Our intuition tells us that both doors should see their probabilities go up: door three, because it might have been opened but was not; door five, because it was part of an equiprobable ensemble that has decreased in size. Bayes' Theorem confirms our intuitions. We compute

$$P(C_3|M_5) = \frac{P(C_3)P(M_5|C_3)}{P(C_3)P(M_5|C_3) + P(C_4)P(M_5|C_4)} = \frac{\frac{1}{4}(1)}{\frac{1}{4}(1) + \frac{3}{8}\left(\frac{1}{2}\right)} = \frac{4}{7},$$

and obtain probability vector

$$\left(0,\ 0,\ \frac{4}{7},\ \frac{3}{7},\ 0\right).$$

Remarkably, our arguments to this point are already enough to justify the correct solution to Progressive Monty. Consider the strategy in which we switch at the last minute (SLM). That is, we will stick with our initial choice until only two doors remain, and then we will switch. Our initial choice has probability $\frac{1}{n}$. Since the other doors are equiprobable, this probability will not change so long as we keep it as our choice. At the moment when only two doors remain, the other door will have probability $\frac{n-1}{n}$. That is the probability that we win with SLM.

We also know that there will never be a moment in the game when a door has a probability smaller than $\frac{1}{n}$. Thus, at the moment when only two doors remain it is impossible to produce a door with probability greater than $\frac{n-1}{n}$. This shows that SLM is optimal.

Very nice. A full, rigorous proof that SLM is, in fact, *uniquely* optimal can be found in the book by Rosenhouse **[13]**. You might also wonder what can be said about other strategies. For example, what if we are playing with fifty doors and we are absolutely determined to switch exactly seven times during the game? What is our best strategy? A consideration of such questions can be found in the paper by Lucas and Rosenhouse [**8**]. Progressive Monty receives further attention in one paper by Paradis, Viader, and Bibiloni [**11**] and another by Rao and Rao [**12**]. For a variation on the problem, see the paper by Zorzi in this issue of the MAGAZINE.

It would seem that a bit of clear thinking can steer us through even the densest of Monty-inspired forests. Once our intuition has been tuned to what is important, it is not so difficult to ferret out the correct answer.

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Modeling a Diving Board

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doi:10.4169/002557009X478364

The *beam equation* is a classic fourth-order partial differential equation used to describe the transverse displacement of a vibrating beam. One may encounter it in an introductory course on boundary value problems or mathematical physics. One particular type of beam modeled with this equation is the *cantilever*, which is a long, thin, rectangular bar with one end clamped and the other end free to move up and down [**3**]. Cantilever beams appear in a variety of applications, including bridge construction [**13**], aircraft wing design [**1**], architecture [**9, 11**], microwave imaging [**15**], temperature measurement [**19**], and virus detection [**7**].

This paper came about as the result of an undergraduate Honors Thesis project at Ball State University [**16**]. The goal of the thesis was to address the question: Since we can think of a diving board as a cantilever, how well does the beam equation model a diving board? To answer this question, we created a simple experimental set-up consisting of a two-meter stick with one end clamped to a table and the free end set in motion. Vertical displacements at tape-marked positions along the beam were captured with a digital video camera, analyzed with the freely available World-in-Motion physics demonstration software [**4**], and compared to a model based on the standard beam equation. As is often the case with models of real-world phenomena, our initial model failed to match the measured data. Using the wave equation as a model [**2**], we first introduced a damping term to improve results and then a forcing term to arrive at a reasonable model.

Before explaining our experiment, we first develop a model for a cantilever beam, based on the beam equation, that includes terms for damping and forcing. For completeness, since the beam equation is usually presented in textbooks as either an example or homework problem with little or no motivation, we summarize a derivation of the beam equation. The derivation, which we have not found in any other textbooks in this form, is similar to standard arguments used by many authors to derive the heat equation or wave equation and relies only on Newton's Second Law and the ideas of *bending moment* and *shearing force*, which will be defined below.

To verify our model, we compare our measured data with our model using *Mathematica*'s new `Manipulate` command. This comparison incorporates all of our investigations into a single model whose parameters can be varied in *real time* to study each of our original models. The numerical computations and graphs included in the paper were all done with *Mathematica 6.0*. A *Mathematica* notebook used to manipulate the model can be found at the MAGAZINE website. (A free *Mathematica Player* is available to permit anyone to view the notebook.) We also offer an appendix that details the analytic solution to the relevant boundary-value problem.

A model for a cantilever beam

Consider a horizontal beam parallel to the x-axis with clamped left end at $x = 0$ and free right end at $x = L$, as shown in FIGURE 1.

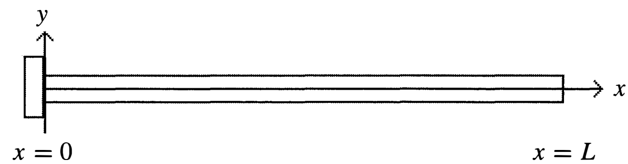

Figure 1 A cantilever beam, fixed at the left end and free to vibrate at the right end

Assume the beam is homogeneous, with uniform rectangular cross section (perpendicular to the x-axis) at any point x along the beam. Take the x-axis to be the initial axis of symmetry of the beam, assume that the beam is in rotational equilibrium, and suppose that the only forces acting on the beam are vertical and lie in the xy-plane. At any time t and at any point x along the beam, forces acting on the beam will displace the beam vertically by an amount $y(x, t)$, deforming the beam's axis of symmetry to form a curve in the xy-plane known as the *elastic curve* [**17**]. These forces will also produce *bending moments* in cross sections of the beam. A *bending moment* $M(x, t)$ in the cross section at any point x along the beam at time t is a torque exerted about a horizontal axis in the cross section [**18**]. We will see that the bending moment at any point is proportional to the curvature of the elastic curve.

In order for the beam to remain in rotational equilibrium, forces within the beam must counter the bending moment at each point. This type of force, known as a *vertical shear* and denoted $V(x, t)$, can be thought of as the force exerted by one side of the beam on the other, about a cross section at any point x along the beam at time t. [**6, 18**]. Note that bending moment and vertical shear vary from point to point along the beam [**6**].

To find a differential equation satisfied by $y(x, t)$, let's examine the forces on a small piece of beam between x and $x + \Delta x$ at time t. Suppose the only forces on this piece of beam are due to its weight, damping from air resistance, and vertical shear at each end. To incorporate damping into the beam equation, we can use Stokes' Law [**8**], which says that for a small sphere traveling slowly through fluid (such as air), the drag force on the sphere is proportional to the sphere's radius and velocity [**5**]. For simplicity, we assume the damping on a small piece of the beam obeys a law similar to Stokes' Law. It follows from Newton's second law that

$$\text{Total Force} = \text{Net Shear Force} + \text{Drag Force} + \text{Weight},$$

that is,

$$m\Delta x \frac{\partial^2 y}{\partial t^2} = V(x, t) - V(x + \Delta x, t) - \gamma \Delta x \frac{\partial y}{\partial t} - m\Delta x g, \tag{1}$$

where m is the mass per unit length of the beam, $\gamma \geq 0$ is a proportionality constant, and g is the acceleration due to gravity. Dividing (1) by $m\Delta x$ and letting $\Delta x \to 0$, we find that

$$\frac{\partial^2 y}{\partial t^2} = -\frac{1}{m}\frac{\partial V}{\partial x} - \frac{\gamma}{m}\frac{\partial y}{\partial t} - g. \tag{2}$$

We really want an equation that only involves the displacement $y(x, t)$. To eliminate the $V(x, t)$ term from (2), we next look at a relationship between vertical shear and bending moment that was established in 1851 by German engineer J.W. Schwedler (1823–1894) for use in his analysis of bridge trusses [**12, 21**]. For beams that are in rotational equilibrium [**17**],

$$V(x, t) = \frac{\partial M}{\partial x}. \tag{3}$$

Differentiating (3) with respect to x, it follows that (2) can be written as

$$\frac{\partial^2 y}{\partial t^2} = -\frac{1}{m}\frac{\partial^2 M}{\partial x^2} - \frac{\gamma}{m}\frac{\partial y}{\partial t} - g. \tag{4}$$

The final relationship we need to get to an equation involving only $y(x, t)$ comes from studies of the bending of beams by Jacob Bernoulli (1654–1705), Leonhard Euler (1707–1783), and Charles-Louis Navier (1785–1836) [**12, 21**]. Bernoulli and Euler observed that, at any point along the beam, the bending moment is proportional to the reciprocal of the radius of curvature of the beam's elastic curve and Navier determined the correct proportionality constant. In particular,

$$M(x, t) = EI\frac{\partial^2 y}{\partial x^2}\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{-3/2}, \tag{5}$$

where E is the *modulus of elasticity* of the beam material and I is the *moment of inertia* of the rectangular cross section of the beam at x with respect to a horizontal line passing through the center of mass of this cross section. The quantity EI, known as the *flexural rigidity*, can be assumed to be constant, due to the uniformity of the beam. If the beam bends only slightly, so that $\partial y/\partial x$ is close to zero, (5) can be approximated by the linear equation

$$M(x, t) = EI\frac{\partial^2 y}{\partial x^2}. \tag{6}$$

Differentiating (6) twice with respect to x and using the fact that the flexural rigidity EI is constant, (4) becomes

$$\frac{\partial^2 y}{\partial t^2} = -\frac{EI}{m}\frac{\partial^4 y}{\partial x^4} - \frac{\gamma}{m}\frac{\partial y}{\partial t} - g. \tag{7}$$

Finally, replacing $\sqrt{EI/m}$ and γ/m in (7) with the constants c and k, respectively, we arrive at the *beam equation with forcing and damping*,

$$\frac{\partial^2 y}{\partial t^2} = -c^2\frac{\partial^4 y}{\partial x^4} - k\frac{\partial y}{\partial t} - g; \quad 0 < x < L,\ t > 0. \tag{8}$$

To complete our model, we need to impose appropriate boundary conditions and initial conditions. For a cantilever beam these are [**14**]

$$y(0, t) = 0, \quad t > 0, \qquad \frac{\partial y}{\partial x}(0, t) = 0, \quad t > 0, \tag{9}$$

$$\frac{\partial^2 y}{\partial x^2}(L, t) = 0, \quad t > 0, \qquad \frac{\partial^3 y}{\partial x^3}(L, t) = 0, \quad t > 0, \tag{10}$$

$$y(x, 0) = f(x), \quad 0 < x < L, \qquad \frac{\partial y}{\partial t}(x, 0) = g(x), \quad 0 < x < L. \tag{11}$$

Boundary conditions in (9) tell us that there is zero displacement and zero slope at the clamped left end of the beam. For the free right end, the boundary conditions in (10) specify no bending moment or shear force at that end, as in (3) and (6). Initial displacement and velocity at each point of the beam are given in (11). We assume that functions $f(x)$ and $g(x)$ are *sectionally smooth* on $[0, L]$, meaning that each function has at most a finite number of removable jumps, discontinuities, and corners, with the function and its derivative continuous between such points [**14**].

Together (8)–(11) define an initial-value boundary-value problem, which we can solve analytically using the standard technique of *separation of variables* [**3, 10, 14, 16**]. For given initial data f and g, the solution is a superposition of perhaps infinitely many functions of x, each progressing in time in a different way. Readers familiar with the method are encouraged to try, though the details are cumbersome. We summarize the results here and refer readers to our appendix, available at the MAGAZINE website.

The solution to (8)–(11) is given by

$$y(x,t) = v(x) + \sum_{n=1}^{\infty} X_n(x)\left(\exp\left(\frac{-kt}{2}\right)[A_n\cos(\mu_n t) + B_n\sin(\mu_n t)]\right) \tag{12}$$

where it takes some doing to explain all the symbols. To capture any particular cantilever beam at any moment, this analytic solution tell us to superimpose certain functions, $X_n(x)$, known as the *eigenfunctions* for this problem. The first three are shown in FIGURE 2. As with the well-known vibrating string, the term in (12) with the least spatially oscillatory eigenfunction vibrates the least quickly in time.

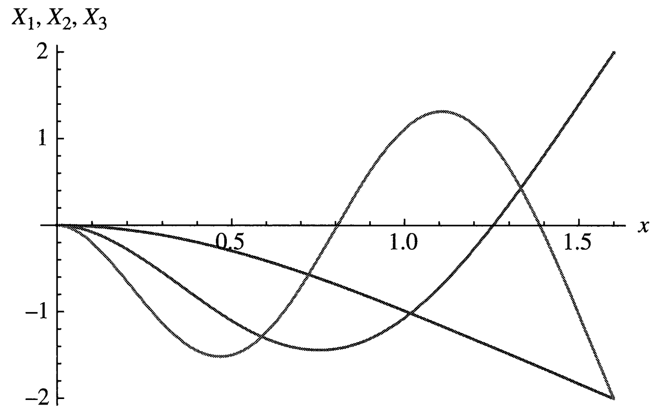

Figure 2 The first three eigenfunctions for the cantilever beam

The portion of the solution with no time dependence is

$$v(x) = -\frac{g\,L^2\,x^2}{4\,c^2} + \frac{g\,L\,x^3}{6\,c^2} - \frac{g\,x^4}{24\,c^2}, \tag{13}$$

which we call the *steady-state solution*. The eigenfunctions are

$$X_n(x) = \cos\alpha_n x - \cosh\alpha_n x - \frac{\cosh\alpha_n L + \cos\alpha_n L}{\sinh\alpha_n L + \sin\alpha_n L}(\sin\alpha_n x - \sinh\alpha_n x), \tag{14}$$

where α_n is the nth positive root of

$$\cos(\alpha L) + \operatorname{sech}(\alpha L) = 0. \tag{15}$$

Notice that for large αL, $\operatorname{sech}(\alpha L)$ is very close to zero, so for large positive integers n, α_n satisfies

$$\alpha_n \approx \frac{(2n+1)\pi}{2L}, \tag{16}$$

which is useful for finding the α_n values numerically.

The angular frequency for the oscillating coefficients of the nth eigenfunction is given by

$$\mu_n = \frac{1}{2}\sqrt{4\alpha_n^4 c^2 - k^2}, \tag{17}$$

requiring that $0 \leq k < 2c\alpha_n^2$. The coefficients A_n and B_n are found via

$$A_n = \frac{\int_0^L (f(x) - v(x))X_n(x)dx}{\int_0^L X_n(x)^2 dx} \tag{18}$$

and

$$-\frac{A_n k}{2} + B_n \mu_n = \frac{\int_0^L g(x)X_n(x)dx}{\int_0^L X_n(x)^2 dx}. \tag{19}$$

Observe that (12) reduces to the *nondamped* case or *no forcing* case by setting k or g equal to zero, respectively.

Verifying our model

A wooden ruler of length two meters represents our physical diving board. It is clamped to one side of a table with a C-clamp and pulled down on the free end as shown in FIGURE 3. Fifteen markings along the ruler, positioned approximately 0.1 meters apart, are analyzed using a video camera attached to a computer with World-in-Motion software to determine their displacement from the x-axis 30 times per second after the free end is released [4]. The software delivers the data as ordered pairs, mea-

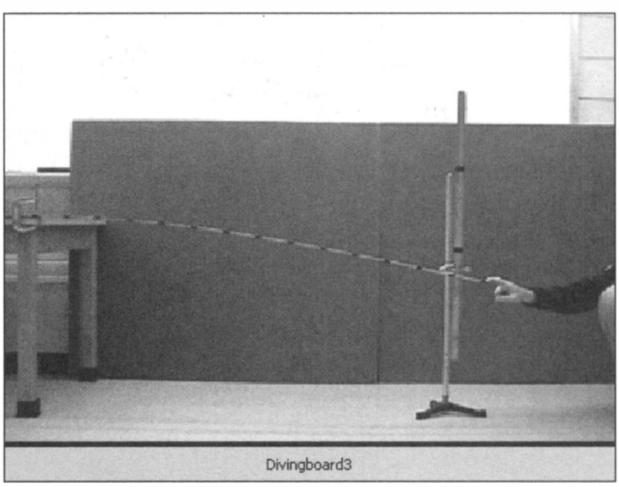

Figure 3 Experimental diving board setup

sured to six significant figures, relative to an origin we set at the left-most tape mark on the beam.

To compare the actual data with our theoretical model, we must determine the model parameters and initial conditions. After clamping the two-meter stick to the table, the beam has length $L = 1.6$ meters. A standard way to model the initial position of a beam uses some simple function, such as $f(x) = 0$ for a horizontal beam or $f(x) = x^2$ for beam curved upward [**3**]. Since our beam curves slightly downwards with origin at the left-most tape mark, an appropriate choice of initial position function is a quadratic polynomial found via a least-squares fit. We calibrate our beam as

$$f(x) = ax - bx^2, \quad 0 \leq x \leq L, \tag{20}$$

with $a = -0.0341084$ and $b = 0.0808230$. For initial velocity, we simply take

$$g(x) \equiv 0, \quad 0 \leq x \leq L. \tag{21}$$

We still need a value for the constant c, as well as for k and g if we wish to include damping and forcing. Let's start by simplifying our model to one without forcing or damping and try to estimate c from its definition in terms of the beam's modulus of elasticity E, moment of inertia I, and mass per unit length m, namely

$$c = \sqrt{\frac{EI}{m}}. \tag{22}$$

The ruler is made of some unknown hardwood, with a mass of 0.3014 kg, overall length of two meters, and rectangular cross- section of width $b = 2.625$ cm and height $d = 0.76$ cm. Hardwoods such as white oak have an elasticity of 1.7×10^6 psi or about 1.2×10^{10} N/m^2 [**20**], and for a beam with a rectangular cross section of width b and height d,

$$I = \frac{1}{12} bd^3, \tag{23}$$

so (22) and (23) suggest $c = 8.7$ m^2/sec as a guess.

Knowing the beam's length L allows us to find the α_ns numerically from (15), using (16) for initial values. Once the α_ns are known, we can use our choices of c, k, g, along with $f(x)$ and $g(x)$ from (20) and (21) in (18) and (19) to compute the coefficients A_n and B_n for our solution given by (12). Note that with k and g set equal to zero, it follows from (13) and (17) that $v(x) \equiv 0$ and $\mu_n = \alpha_n^2 c$ for all $n \geq 1$. Since $g(x) \equiv 0$, $B_n = 0$ for all $n \geq 1$. For the A_ns, we can find the first seven coefficients numerically, but run into trouble for larger integers because the $\cosh \alpha_n x$ and $\sinh \alpha_n x$ terms in (14) grow rapidly for large values of $\alpha_n x$. Graphically comparing the seventh partial sum of (12) to $f(x)$ at $t = 0$, we judge it to be sufficient for our model.

FIGURE 4 compares the data and our model without damping or forcing at the right-most tape mark at the free end of the beam. (Due to a time delay in releasing the beam before the camera began filming, we choose time $t = 0$ to correspond to the fifth set of recorded data points.) Unfortunately, our choice of c based on (22) leads to a poor match between model and measured data.

Since we don't really know what type of wood makes up the ruler, another way to estimate c is to use the fact that, with no damping or forcing, the only place that c appears in our model is in the $\cos(\alpha_n^2 ct)$ terms of (12) via (17). It follows that the terms in our model will have periods P_n given by

$$P_n = \frac{2\pi}{\alpha_n^2 c}, \tag{24}$$

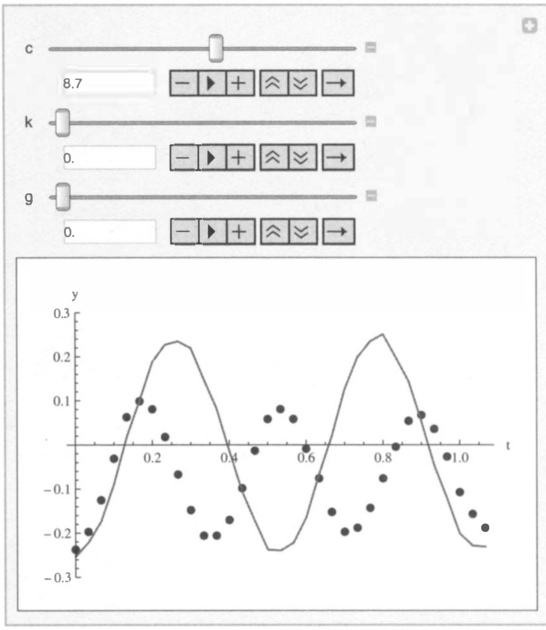

Figure 4 Plot of beam displacement y (m) versus time t (sec) at the free end of the beam with no damping or forcing terms in the model. The solid curve represents model data and the dots represent experimental data with $c = 8.7$ m^2/sec, $k = 0$ sec^{-1}, and $g = 0$ m/sec^2.

for $n = 1, 2, \ldots$. Looking at the beam displacement data at the free end and using the fact that consecutive displacement data values are $1/30$ second apart, we see that the beam's first three periods in time are approximately $11/30$ sec, $10/30$ sec, and $11/30$ sec, respectively. The average of these three periods is $16/45$ sec. Since F_1 is the largest coefficient in magnitude in our model, and thus will have the most influence, we use period P_1 to find our second estimate for c. With $\alpha_1 = 1.17194$ and $P_1 = 16/45$ sec in (24), it follows that c should be about 12.87 m^2/sec. (Another way to find c experimentally that requires no knowledge of the beam's composition appears in the appendix to this paper at the MAGAZINE website.)

FIGURE 5 compares the data and our model with our second choice of c, again without damping or forcing, at the free end of the beam. Periods of the model now match those of the data, but our model does not predict the same amplitude as the measured data over time. In fact, unlike our model, the beam's amplitude decreases slightly over time, which suggests that at least damping and possibly forcing should be incorporated into the model. Also note that this method to determine c doesn't take all terms in $y(x, t)$ into account, which could be a factor, as each term's period depends on an α_n value, which changes for each n.

Introducing forcing and damping increases the complexity of our model. From (13) we see that c and g appear in the steady-state solution $v(x)$, and from (17), it follows that k occurs along with c in the coefficients μ_n. Thus, we need to find a way to choose values for c, k, and g that takes this into account when computing A_n and B_n with (18) and (19). With the advent of the new `Manipulate` command in *Mathematica 6.0*, we can include *every* term in our partial sum solution and dynamically estimate all three constants c, k, and g at once. Using `Manipulate`, we set up the model with c, k, and g as variables that can be changed via sliders in *real time* to see how the solution is affected. (This is how FIGURES 4 and 5 were actually generated.) The comparison of

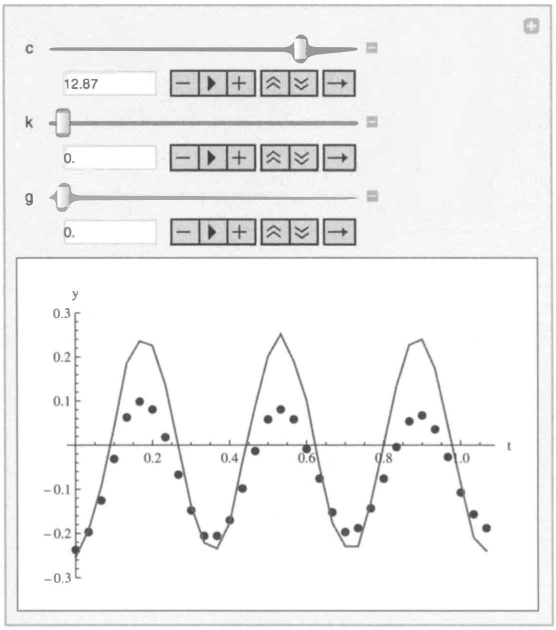

Figure 5 Plot of beam displacement $y(L)$ (m) versus time t (sec) at the free end of the beam with no damping or forcing terms in the model. The solid curve represents model data and the dots represent experimental data with $c = 12.87$ m^2/sec, $k = 0$ sec^{-1}, and $g = 0$ m/sec^2.

the model solution to actual data is done graphically, with values of c, k, and g chosen to get a reasonable graphical fit.

Keeping $c = 12.87$ m/sec and $g = 0$ m/sec^2, we obtain a better graphical fit with $k = 1.81$ sec^{-1}, as seen in FIGURE 6. Introducing damping to our model provides a significant improvement, reducing the beam's amplitude over time for better agreement graphically, but the measured data values consistently lie below the model values, especially at the beam's right end. To account for this, we introduce forcing to the model by choosing $g = 9.8$ m/sec^2 and adjusting c or k as needed. FIGURE 7 shows the beam's displacement over three periods at the free end with damping and forcing included in the model. With $c = 12.87$ m^2/sec, $k = 0.94$ sec^{-1}, and setting g to the known value of 9.8 m/sec^2, we find an MSSE (mean of the sum of the squares for error) of 0.010 m between the model and measured data. As a check, and also to see if we can do any better, using *Mathematica*'s `FindMinimum` command with these initial choices for c and k, we find numerically that $c = 12.82$ m^2/sec and $k = 1.01$ sec^{-1}, yielding essentially the same error as our graphical fit. FIGURE 8 shows the beam's displacement over three periods at all fifteen points along the beam with these new choices of c and k.

Conclusion

Using a simple experimental setup (wooden two-meter stick, clamp, tape, digital video camera, and computer software), we have been able to show that the beam equation can be used to model a vibrating cantilever beam. As is often the case in modeling real-world phenomena, we had to revise our initial model to get a model that matches reality!

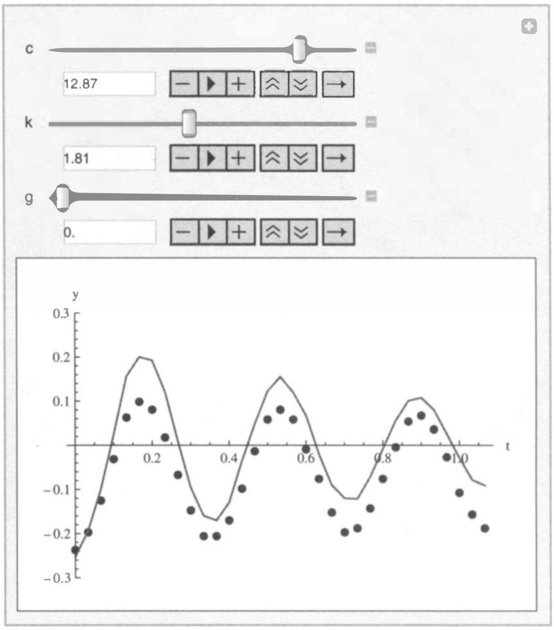

Figure 6 Plot of beam displacement y (m) versus time t (sec) at the free end of the beam with a damping term added to the model. The solid curve represents model data and the dots represent experimental data with $c = 12.87$ m^2/sec, $k = 1.81$ sec^{-1}, and $g = 0$ m/sec^2.

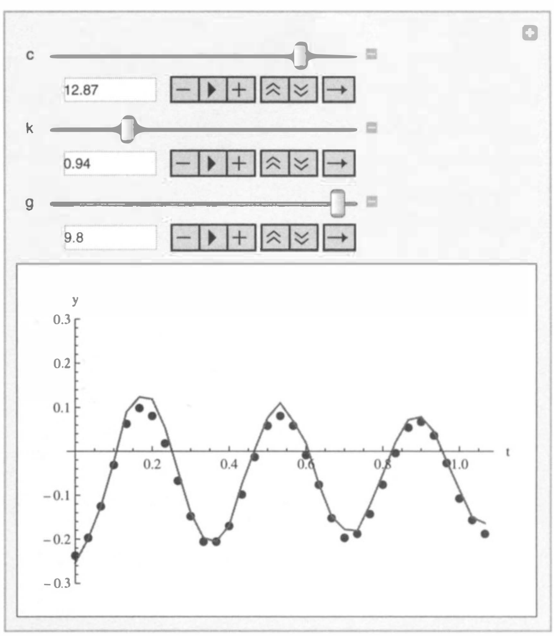

Figure 7 Plot of beam displacement y (m) versus time t (sec) at the free end of the beam with a damping term and a forcing term added to the model. The solid curve represents model data and the dots represent experimental data with $c = 12.87$ m^2/sec, $k = 0.94$ sec^{-1}, and $g = 9.8$ m/sec^2.

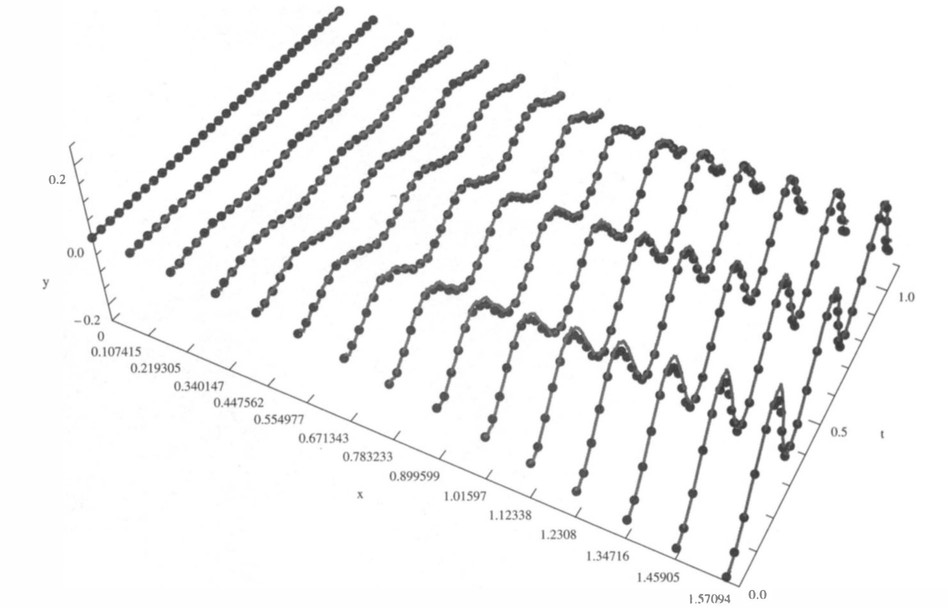

Figure 8 Plot of beam displacement y (m) versus time t (sec) at the free end of the beam with a damping term and a forcing term added to the model. The solid curve represents model data and the dots represent experimental data with $c = 12.84$ m^2/sec, $k = 1.01$ sec^{-1}, and $g = 9.8$ m/sec^2.

In addition to the experiment outlined, other questions could be addressed. Would the beam equation model an *actual* diving board? What about adding in an impulse at the free end to model a person springing off the board? How about other physical systems for which the beam equation with different boundary conditions would be an appropriate model, such as a bridge span with supports at both ends? Are there other forcing terms that would be more appropriate—for example, how about a forcing term that is a function of the distance from the clamped end? Would such a forcing term work better mathematically? Do forcing terms such as these make sense physically?

Acknowledgment. All equipment for our experiments, as well as lab space, and the Wave-in-Motion software were provided by the Department of Physics and Astronomy at Ball State University. Without their generous support, this project would not have been possible. We would also like to thank the referees for their suggestions, especially the method for finding c experimentally outlined in the appendix to this paper.

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Letter to the Editor: Correction to "Fibonacci Clock"

doi:10.4169/193009809X468959-r

To the Editor:

I would like to correct two statements from my note, "Long Days on the Fibonacci Clock," published in Vol. 82, no. 2 of the MAGAZINE.

The first case examined in the note is where 5 is a quadratic residue of p. Thus, $\sqrt{5}$ is an element of $\mathbf{F}_p$, meaning there is no reason (a priori) that $\mu = (1+\sqrt{5})/2$ and $\bar{\mu} = (1-\sqrt{5})/2$ should have the same order. The point here is that they are not really conjugate in the way I insinuated through my choice of notation. Indeed, loosely speaking, in half the cases the orders may differ by a factor of 2, as can be seen as follows.

Note, first, that $\mu\bar{\mu} = -1$. If -1 is not a residue of p, then either μ is a residue and $\bar{\mu}$ is not, or vice versa. But then the eigenvalue that is a quadratic residue is equal to λ^2 and its order will be half the order of the other eigenvalue. The condition for -1 to be a nonresidue is that $p \equiv 3 \pmod 4$. Combining this with the condition for 5 to be a residue of p and assuming that p is odd, leaves us with the condition that $p \equiv 11 \pmod{20}$ or $p \equiv 19 \pmod{20}$.

Even so, Proposition 1 remains true, as you need both μ^k and $\bar{\mu}^k$ to be 1 for the sequence to cycle back on itself.

In Proposition 2, I showed that $\mu^{2(p+1)} \equiv 1 \pmod p$, and that $\mu^{p+1} \equiv -1 \pmod p$, which shows that the order of μ is not going to be $p+1$, but did not check that the order of μ might be some other divisor of $2(p+1)$, a phenomenon we essentially saw in the first case, where 5 is a quadratic residue of p. What we do know is that the period is either $2(p+1)$ or $2(p+1)$ divided by an odd number, and such cases occur.

I am grateful to Erich Badertscher for his help with the corrections.

Yours sincerely,

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NOTES

Self-Curvature Curves

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doi:10.4169/002557009X478373

The signed curvature function for a plane curve gives a quantitative way to describe how much and in which direction the curve bends. For the graph $y = y(x)$, the signed curvature function is known to be

$$\kappa(x) = \frac{y''(x)}{\left(1 + y'(x)^2\right)^{3/2}}. \tag{1}$$

Gray [**1**] or Thorpe [**2**] explain more about the significance of the signed curvature function and how it is formulated. As an example that leads to our topic, consider the curve $y = \cos(x)$. Its curvature function is

$$\kappa(x) = -\frac{\cos(x)}{\left(1 + \sin^2(x)\right)^{3/2}}.$$

Note in FIGURE 1 the general similarity between the cosine curve and the graph of its curvature function, especially after reflecting the graph of the curvature function about the x-axis. A natural question is whether there are any curves that really *are* the same as the graphs of their curvature functions; or, considering the example in FIGURE 1, whether there are any curves that are the same as the graphs of their curvature functions up to a possible vertical reflection and dilation. If such curves exist, call them *self-curvature curves.*

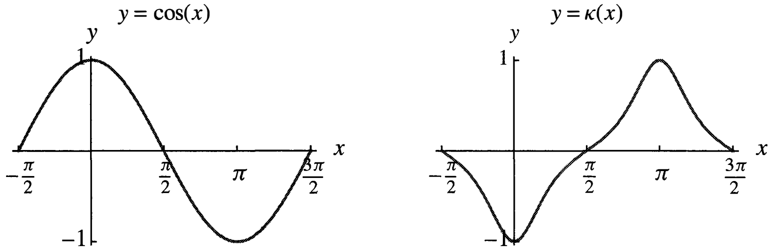

Figure 1 The graph $y = \cos(x)$ and the graph of its curvature function

The horizontal line $y = 0$ is a trivial example of such a curve since the curvature of a straight line is 0. Are there any nontrivial examples? From (1), if the graph of a function is to match the graph of its curvature function up to a constant factor, then

$$\frac{y''(x)}{\left(1+y'(x)^2\right)^{3/2}} = ay(x)$$

for some $a \neq 0$. Studying self-curvature curves is equivalent to analyzing this ordinary differential equation.

Under what circumstances should we expect solutions? The existence and uniqueness theorem for ODEs [**3**, p. 127] says that the initial condition problem

$$y''(x) = f\big(x, y(x), y'(x)\big), \quad y(x_0) = y_0, \quad y'(x_0) = p_0 \tag{2}$$

has a unique solution, defined on some open interval containing x_0, if the partial derivatives of f with respect to y and y' are continuous around (x_0, y_0, p_0). Our differential equation satisfies the partial derivative constraints, so there is a self-curvature curve for each set of initial conditions $y(x_0) = y_0$, $y'(x_0) = p_0$. Further analysis of the differential equation shows that nontrivial self-curvature curves fall into two main categories, characterized by whether the domain of the solution is the whole line or a compact interval. If the domain is the whole line, we will call the function a *periodic* self-curvature curve. The category where the domain is a compact interval splits into two types, the *nonzero* and the *monotonic* self-curvature curves. FIGURE 2 shows the three basic types. Notice in each case how the curvature varies with the curve's proximity to the x-axis. Let us proceed to work out the details.

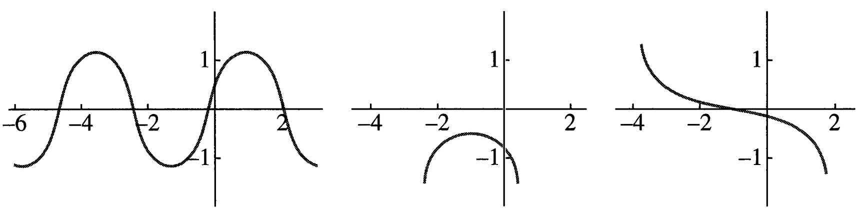

Figure 2 Three basic shapes of self-curvature curves

Qualitative analysis Consider the second order initial condition problem:

$$\frac{y''}{\left(1+y'^2\right)^{3/2}} = ay, \quad y(x_0) = y_0, \quad y'(x_0) = p_0. \tag{3}$$

Solutions to this equation have symmetries we can verify from the uniqueness theorem, once we know what to look for.

Suppose the solution has an x-intercept—where the curvature must necessarily vanish—and name it x_1 so that $y(x_1) = 0$. In the examples shown, there seems to be symmetry about every such point, so let $\tilde{y}(x) = -y(-x + 2x_1)$. It satisfies the same differential equation as $y(x)$,

$$\frac{\tilde{y}''(x)}{\left(1+\tilde{y}'(x)^2\right)^{3/2}} = \frac{-y''(-x+2x_1)}{\left(1+y'(-x+2x_1)^2\right)^{3/2}} = -ay(-x+2x_1) = a\tilde{y}(x),$$

and also the same initial conditions at x_1:

$$\tilde{y}(x_1) = -y(-x_1 + 2x_1) = -y(x_1) = 0 = y(x_1)$$
$$\tilde{y}'(x_1) = y'(-x_1 + 2x_1) = y'(x_1).$$

Since the solution must be unique, the two functions are equal. We conclude that $y(x) = -y(-x + 2x_1)$, confirming that y is symmetric about the point $(x_1, 0)$.

This same symmetry implies that if there are at least two distinct zeroes to the solution, then the solution must be periodic. If x_1 and x_2 are the two zeroes, use the symmetry relation twice to find

$$y(x) = -y(-x + 2x_1) = -\big(-y\big(-(-x + 2x_1) + 2x_2\big)\big) = y\big(x + 2(x_2 - x_1)\big).$$

This shows that if x_1 and x_2 are two zeroes of y with no other zero between them, then y is periodic with period $2|x_2 - x_1|$.

The middle graph in FIGURE 2 suggests another symmetry. The reader can use a similar argument to show that if the solution has a critical point at x_1, then $y(x) = y(-x + 2x_1)$, which means that the solution is symmetric about the vertical line through that critical point. Also, if there are two distinct critical numbers, x_1 and x_2, with no other critical number between them, the solution must be periodic with period $2|x_2 - x_1|$.

Implicitly defined solutions Although we cannot solve the initial condition problem explicitly, we can solve it implicitly. To facilitate integration, multiply both sides of $y''/\left(1 + y'^2\right)^{3/2} = ay$ by y' to get

$$\frac{y'y''}{\left(1 + y'^2\right)^{3/2}} = ayy'.$$

Antidifferentiate, using an easy substitution, and solve for y' to find

$$y' = \pm\frac{\sqrt{4 - (2c + ay^2)^2}}{2c + ay^2}. \tag{4}$$

This new ODE is separable; perform one more integration to see that

$$F(y) = \pm(x - x_0), \tag{5}$$

where

$$F(y) = \int_{y_0}^{y} \frac{2c + a\psi^2}{\sqrt{4 - (2c + a\psi^2)^2}}\, d\psi. \tag{6}$$

The choice of the sign in (5) is determined by the initial height y_0 and slope p_0.

The integrand in F is continuous as long as we keep the radicand in the denominator positive. Moreover, we can determine that the improper integral converges when y is a value of ψ that makes the denominator zero. From (4), those values correspond to points where the solution has horizontal tangents. The domain of F is included in the set of y-values such that

$$-2 \le 2c + ay^2 \le 2 \text{ or } -2 - 2c \le ay^2 \le 2 - 2c. \tag{7}$$

The values of y where the denominator in (4) is zero correspond to points on the curve with vertical tangents. The only way a continuous curve could pass through such a point and still remain the graph of a function is if the concavity changes at that point. Since $y''/(1 + y'^2)^{3/2} = ay$, this can only happen if y changes sign, that is, passes through zero. That means if a vertical tangent occurs with nonzero height, the curve is defined on only one side of that point.

There is one other equation we will use—the relation between the constant of integration c in (4) and the initial conditions y_0 and p_0. Evaluate equation (4) at the initial x-value and solve for c.

$$c = -\frac{1}{\sqrt{1+p_0^2}} - \frac{ay_0^2}{2} \tag{8}$$

Combine what we know about vertical tangents in (4), the possible y-values in (7), and the initial condition information in (8) to see that different combinations of values for a and c give three categories of nontrivial self-curvature curves.

Monotonic curves If $a > 0$ and $c > -1$, then (7) reduces to $|y| \le \sqrt{\frac{2-2c}{a}}$. This interval of y-values will contain the two points of vertical tangency $y = \pm\sqrt{-2c/a}$. Solving (8) for y_0^2 shows that y_0, the initial value of the solution, lies between the two points of vertical tangency so the range of the solution y is $\left[-\sqrt{-2c/a}, \sqrt{-2c/a}\right]$. With these values of y and (4) we find that y' is never zero so that $y(x)$ must be monotonic. The corresponding domain of the solution by (5) is the closed interval $\left[F(-\sqrt{-2c/a}) + x_0, F(\sqrt{-2c/a}) + x_0\right]$ if the initial slope p_0 is positive. If $p_0 < 0$, then the domain is $\left[-F(\sqrt{-2c/a}) + x_0, -F(-\sqrt{-2c/a}) + x_0\right]$.

EXAMPLE. $a = 1, c = -1/\sqrt{2}, x_0 = y_0 = 0, p_0 = 1$. The range of the solution is $\left[-\sqrt[4]{2}, \sqrt[4]{2}\right] \approx [-1.19, 1.19]$ and its domain is $\left[F(-\sqrt{2}), F(\sqrt{2})\right] \approx [-0.703, 0.703]$. Use your favorite numerical differential equations solver and plotter to verify the curve's shape.

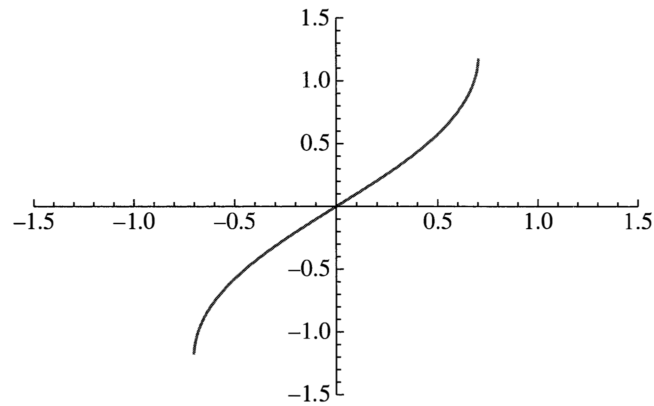

Figure 3 A *monotonic self-curvature curve* with $a = 1$, $c = -1/\sqrt{2}$

Periodic curves If $a < 0$ and $c < 0$, $2c + ay^2$ is strictly negative; by (4), there can be no vertical tangents. Inequality (7) reduces to

$$-\sqrt{\frac{-2-2c}{a}} \le y \le \sqrt{\frac{-2-2c}{a}}.$$

Both endpoints correspond to critical points of the solution. By our previous qualitative analysis, the solution is periodic with period

$$2\left|F\left(\sqrt{\frac{-2-2c}{a}}\right) - F\left(-\sqrt{\frac{-2-2c}{a}}\right)\right|.$$

EXAMPLE. $a = -1$, $c = -1/2$, $x_0 = y_0 = 0$, $p_0 = \sqrt{3}$. The range of the solution is $[-\sqrt{(-2-2c)/a}, \sqrt{(-2-2c)/a}] = [-1, 1]$ and the solution has period $2|F(1) - F(-1)| \approx 5.00$.

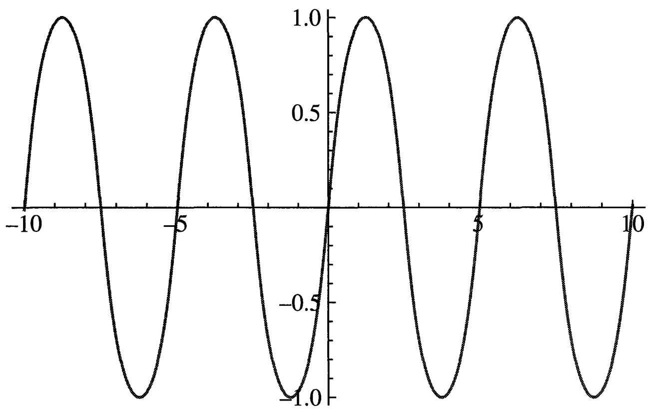

Figure 4 A *periodic self-curvature curve* with $a = -1$, $c = -1/2$

Nonzero curves The remaining two cases are $a > 0$ with $c \leq -1$ and $a < 0$ with $c > 0$. The reader can use similar analyses to see that these lead to what we call the *nonzero* self-curvature solutions. If $a > 0$, by equation (3) the second derivative will have the same sign as the solution, so the curve will always bend towards the x-axis. These are the *innie* self-curvature curves, whereas if $a < 0$, we get the *outie* self-curvature curves. FIGURE 5 shows an innie with $a = 1, c = -2, x_0 = 0, y_0 = 3/2$, $p_0 = \sqrt{15}/7$ and an outie with $a = -1, c = 1, x_0 = 0$, $y_0 = 2$, $p_0 = 0$.

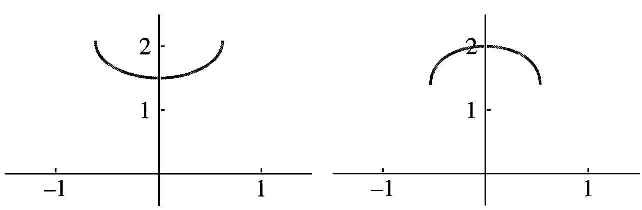

Figure 5 An *innie* on the left with $a = 1$, $c = -2$ and an *outie* on the right with $a = -1$, $c = 1$

In our definition of self-curvature curves, we permitted the curve to match the graph of its curvature function up to a reflection and dilation. Our analysis shows that the reflection was necessary to obtain the periodic self-curvature curves, which are defined on the whole real line. Intuitively, the larger the magnitude of the original function, the more its graph bends, since the original function and its curvature are directly proportional. If the function's value becomes too great, the curve risks bending back onto itself, violating the vertical line test for a function. That is why the monotonic and nonzero self-curvature curves are defined only on compact intervals. Only by having opposite signs for the function and its second derivative, $a < 0$, and starting with a small initial height, $c < 0$, can we keep the function's value small enough so that it is defined on the entire real line.

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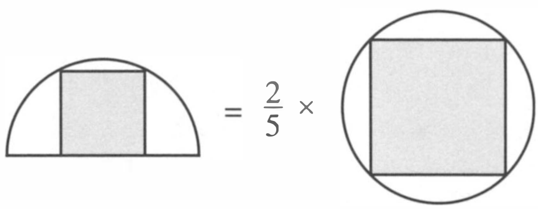
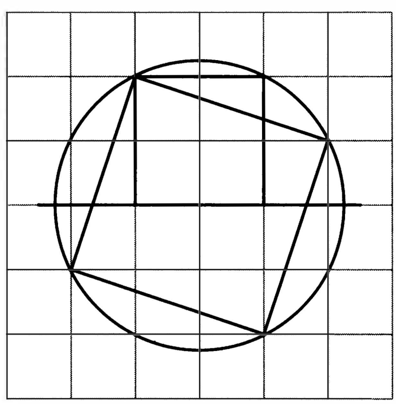
Proof Without Words: Squares in Circles and Semicircles

doi:10.4169/002557009X478382

A square inscribed in a semicircle has 2/5 the area of a square inscribed in a circle of the same radius.



Proof.



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Cars, Goats, π, and e

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doi:10.4169/002557009X478391

People seldom use information as effectively as possible when faced with decisions under uncertain conditions; a famous example is the Monty Hall Problem [**3**]. We examine two particular variations on the problem, derived respectively from papers [**7**] and [**6**], and we show a curious connection between the probability of winning and the numbers π and Euler's number e. That is, players can approximate π and e by playing the variation games repeatedly using the best strategies, in a way very similar to approximating π by dropping Buffon's needle [**1**].

The Monty Hall Problem The problem gets its name from a TV game show, "Let's Make A Deal," hosted by Monty Hall. We summarize as follows:

PROBLEM 1. *There are 3 closed doors: One hides a car and each other one a goat. The game's host knows what is behind each door, but the contestant does not and tries to win the car by choosing the door that hides it. The host asks the contestant to choose a door; then, the host opens a door drawing it randomly from the remaining ones without the car, and asks the player if he/she would like to exchange the chosen door with the still-closed one.*

Is it better for the contestant to change the door or to keep the first he chose?

As detailed in an article in this issue of the MAGAZINE [**5**], the wrong solution is quite common, with people perhaps reasoning as follows: "There are two closed doors left, therefore the probability that the car is behind either one is $1/2$; so there is no reason to change." Of course, the correct choice is always to change. In fact, if the contestant changes, he/she loses if and only if the first chosen door hides the car, and therefore the probability of winning is $2/3$.

Monty Hall and π Let's examine an extension of Monty Hall's problem, originally described in a paper in the Italian journal *Archimede* [**7**]. In this version there are n doors ($n \geq 3$ and n odd), which coincides with the original if $n = 3$. As in Problem 1, only one door conceals a car while each of the others hides a goat. The game's host knows where the car is, but the contestant does not and must guess it to win. The game evolves as follows:

PROBLEM 2. *There are n closed doors, n is odd, and $n \geq 3$.*

1. *The contestant picks a door from among the closed and not yet chosen ones.*
2. *If no closed and not yet chosen door exists, then the game is over and the last chosen door is opened. Otherwise, the game's host opens a door, picking it randomly from among the doors that are closed, not yet chosen, and without the car. Then he asks the contestant if he/she wants to switch to one of the closed and not yet chosen doors.*
3. *If the contestant decides not to change, the last chosen door is opened and the game is over. Otherwise, the game goes back to Step 1.*

What is the best strategy for the contestant?

It is supposed that the number n of doors is odd so that, after the host opens a door, the contestant still has the opportunity to change his last choice. We will now show what the optimal strategy is, and the corresponding probability to win. For convenience, we refer to the probability that a particular door conceals the car as "the probability of the door."

PROPOSITION 1. *In Problem 2, the best strategy for the contestant is to change doors at every opportunity. In this case the probability of winning is*

$$p(n) = \frac{2 \cdot 4 \cdots (n-3)(n-1)}{3 \cdot 5 \cdots (n-2)n}$$

and

$$\lim_{n\to\infty} p(n)^2 \cdot n = \frac{\pi}{2}. \tag{1}$$

Proof. We name the probability of winning as $p(n)$, assuming that the contestant always decides to change. The probability that the first chosen door hides the car is $\frac{1}{n}$. After the host has opened the first door without the car, the probability of each of the $n-2$ unchosen and still closed doors is $\frac{n-1}{n-2}\frac{1}{n}$, because all $n-1$ of the probabilities must add to one. If n were 3, we would be done, concluding that the player who switches wins with probability 2/3. In the general case, it is as if the player has begun a new game with $n-2$ doors, but with the chance of winning increased by a factor of $(n-1)/n$. Thus, $p(n) = \frac{n-1}{n}p(n-2)$, leading to the product formula given. This makes it easy to prove by induction that $p(n) > 1/n$ for all odd $n \geq 3$. Changing is the best strategy, because if the contestant does not switch (Step 3), then the probability of getting the prize will, for some k, be

$$\frac{(n-1)(n-3)\cdots(n-2k-1)}{n(n-2)\cdots(n-2k)} \frac{1}{(n-2k-2)}$$
$$< \frac{(n-1)(n-3)\cdots(n-2k-1)}{n(n-2)\cdots(n-2k)} p(n-2k-2) = p(n).$$

From the Wallis formula $\lim_{n\to\infty}\left(\frac{2\cdot4\cdots(n-3)(n-1)}{3\cdot5\cdots(n-4)(n-2)}\right)^2/n = \pi/2$ [2], we deduce the desired result (1). ∎

Monty Hall and *e* Our next version of the Monty Hall Problem with n doors is similar to one described in the *Monthly* [6]. The doors are replaced by boxes, because we need to shuffle them. Moreover, $n \geq 3$ and n is not necessarily odd. Only one box hides a prize, while all the others are empty. The host knows where the prize is, but the contestant does not and must guess it to win. The new version of the problem is:

PROBLEM 3. *There are n closed boxes, where n $\geq$ 3. The contestant picks a closed box. If exactly one other closed box remains, the game is over and the last chosen box is opened to see whether the contestant wins the prize. Otherwise:*

1. *The host opens a box at random from among the empty, unchosen, and closed ones.*
2. *Then the host secretly shuffles the closed boxes, except the one just chosen by the contestant.*
3. *Finally, the host asks if the contestant wants to change boxes. If the contestant switches, the game continues from Step 1; otherwise the last chosen box is opened and the game is over.*

As in Problems 1 and 2, the question is to find the optimal strategy for the contestant.

The next proposition proves that, as in Problem 2, the best strategy is always to agree to change. Surprisingly, the corresponding probability of winning involves Euler's number e.

PROPOSITION 2. *In Problem 3, the best strategy for the contestant is always to change the box. In this case the probability of winning is*

$$p(n) = \sum_{i=1}^{n-2} \frac{(-1)^{i-1}}{i!} + \frac{(-1)^n}{(n-2)!n} \tag{2}$$

and

$$\lim_{n\to\infty} p(n) = 1 - \frac{1}{e}. \tag{3}$$

Proof. If the game returns to Step 1 with k closed boxes for the contestant to choose from, let p_k denote the probability of each box. Hence $p_n = \frac{1}{n}$ is the probability of the first chosen box. As before, after the host has opened the first box with no prize, each closed box except that chosen has probability $p_{n-2} = \frac{n-1}{n-2}\frac{1}{n} > \frac{1}{n} = p_n$. Therefore a shrewd contestant changes boxes. (We record for future reference that $p_{n-2} < \frac{1}{n-2}$.) Then, if $n > 3$, the host opens another empty box, randomly chosen from the $n-3$ available, and shuffles the remaining closed boxes except for the last chosen. Thus, each unchosen and closed box has probability

$$p_{n-3} = \frac{1-p_{n-2}}{n-3} = \frac{1}{n-3} - \frac{1}{(n-3)(n-2)} + \frac{1}{(n-3)(n-2)n} > p_{n-2}.$$

The last inequality is true since

$$p_{n-2} < \frac{1}{n-2} \text{ implies } \frac{1}{n-2} < \frac{1-p_{n-2}}{n-3} = p_{n-3},$$

which shows in turn that $p_{n-3} < \frac{1}{n-3}$. Assuming that $p_k = \frac{1-p_{k+1}}{k}$, by reverse induction we have $\frac{1}{k+1} < p_k < \frac{1}{k}$ and

$$p_k = (k-1)!\left[\sum_{i=k}^{n-2} \frac{(-1)^{i-k}}{i!} + \frac{(-1)^{n-k+1}}{(n-2)!n}\right] \quad \text{if } 1 \le k \le n-3.$$

Therefore, $p_n = \frac{1}{n} < p_{n-2} < \frac{1}{n-2} < \cdots < \frac{1}{k+1} < p_k < \frac{1}{k} < \cdots < p_1$ and the best strategy is to always change the box. We conclude (2) by setting $k = 1$, and (3) follows from the well-known Maclaurin series. ■

In an article by Lucas and Rosenhouse [**4**], we learned of another extension of the Monty Hall Problem where the probability of winning is $1 - \frac{1}{e}$. Their paper identifies even more phenomena with this seemingly ubiquitous probability.

Final remarks It is interesting to observe that, while in the game of Problem 2 $p(n) \approx \sqrt{\frac{\pi}{2}}/\sqrt{n} \to 0$ as $n \to \infty$, in the game of Problem 3 $\lim_{n\to\infty} p(n) = 1 - \frac{1}{e} > 0.632$ is positive. In fact, in contrast to Problem 2, it is no longer true that if at some time the contestant chooses the box with the prize and then moves to another box, then the contestant is doomed to lose. So, this is a much better game for the contestant.

As observed above, it is possible to approximate π and e playing the games from Problems 2 and 3 over and over again. Play with as many doors as practical, always change doors, and record the frequencies of victories. As you may suspect, the formula $2np(n)^2$, derived from Proposition 1, is not the best way to approximate π; for example, it gives $\pi \approx 3.13$ for $n = 101$. The formula $\frac{1}{1-p(n)}$ for e does much better: if $n = 10$, it gives correctly the first six decimal digits of e.

Finally we notice that, while in Buffon's needle problem the appearance of π is not unexpected (π is the average of the possible inclinations of the needle), a geometrical interpretation of Proposition 1 is not evident.

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A Graph Theoretic Summation of the Cubes of the First *n* Integers

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doi:10.4169/002557009X478409

The complete graph K_{n+1} has $n+1$ vertices and $\binom{n+1}{2}$ edges. Iteratively building the complete graph K_{n+1}, introducing vertices one at a time, and counting new edges incident to each new vertex provides a combinatorial proof that $\sum_{i=1}^{n} i = \binom{n+1}{2}$ [**1**].

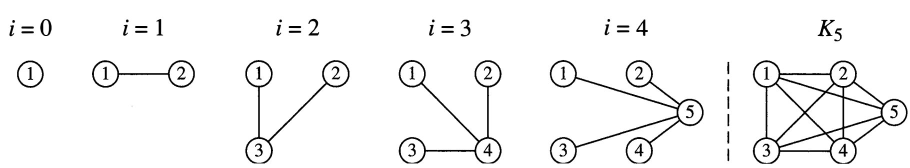

Figure 1 $\sum_{i=1}^{4} i = \binom{4+1}{2}$

Since $\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2$ it seems natural to look for a combinatorial proof that also uses graphs. The relevant graphs turn out to be *bipartite*, meaning that the vertices are partitioned into two sets and edges occur only between vertices from different parts.

Consider the complete bipartite graph $K_{\binom{n+1}{2},\binom{n+1}{2}}$, which contains $2\binom{n+1}{2}$ vertices and $\binom{n+1}{2}^2$ edges. As before, we will introduce new vertices in n stages and count the new edges as they appear. At stage i, we introduce i new vertices to each side of the graph and count the edges incident to these new vertices. Since $\sum_{i=1}^{n} i = \binom{n+1}{2}$ this process enumerates all the edges in $K_{\binom{n+1}{2},\binom{n+1}{2}}$.

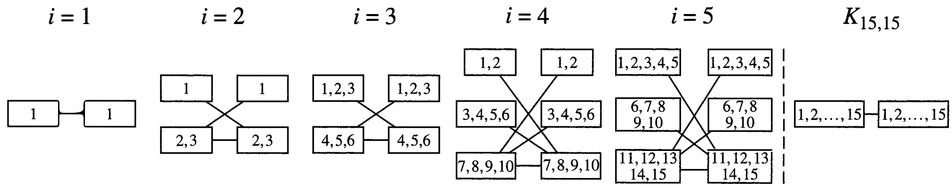

Figure 2 $\sum_{i=1}^{5} i^3 = \binom{5+1}{2}^2$

FIGURE 2 illustrates these stages for $n = 5$. To prevent a deluge of edges in the graph, a complete bipartite graph such as $K_{2,4}$ is represented as [1,2]—[1,2,3,4]. It takes some time to understand the count, so let us walk through the enumeration.

Since the graph is bipartite, new vertices on one side are adjacent only to vertices on the other side. When we consider only the edges among the new vertices, the subgraph $K_{i,i}$ immediately appears, accounting for i^2 edges. It remains to show that these i^2 edges along with the additional edges constructed between new vertices on one side and old vertices on the other side will always total i^3 new edges.

In order to see that we always introduce i^3 new edges at stage i, we partition the new edges into complete bipartite graphs. At stage i, there are already $\binom{i}{2} = \frac{i(i-1)}{2}$ old vertices on each side of the graph. We label the new vertices on each side as $\binom{i}{2}+1, \binom{i}{2}+2, \ldots, \binom{i}{2}+i = \binom{i+1}{2}$. Our method to organize the newly introduced edges into complete bipartite graphs depends upon the parity of i.

When i is odd, the new edges quickly form i disjoint copies of $K_{i,i}$, as follows: For any odd i, we partition the old vertices, which number $\frac{i(i-1)}{2}$, into $\frac{i-1}{2}$ sets of i vertices for each side. Both sets of i new vertices are adjacent to each of the $\frac{i-1}{2}$ sets of i vertices on the other side. This yields $2(\frac{i-1}{2}) = i - 1$ additional copies of $K_{i,i}$. Along with the initial copy of $K_{i,i}$ on only the new vertices, we have i copies of $K_{i,i}$ for a total of i^3 new edges.

When i is even, we have to work a bit harder. For even i, we write the number of old vertices as $\frac{i(i-1)}{2} = i(\frac{i}{2} - 1) + \frac{i}{2}$. This means we can partition the old vertices on each side into $\frac{i}{2} - 1$ sets of i vertices and one set of $\frac{i}{2}$ vertices. This yields $2\left(\frac{i}{2} - 1\right)$ copies of $K_{i,i}$ and two copies of $K_{\frac{i}{2},i}$ for $2\left(\frac{i}{2} - 1\right)i^2 + 2\frac{i}{2}i = i^3 - i^2$ edges. As before, with the original $K_{i,i}$ between the sets of new vertices, the total once again is i^3 new edges.

Since in either case there are i^3 new edges, this demonstrates that $\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2$.

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Elementary Proofs of Error Estimates for the Midpoint and Simpson's Rules

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doi:10.4169/002557009X478418

Usually encountered during a second course in calculus are the

$$\text{Trapezoid Rule,}\quad \int_a^b f(x)\,dx \cong \frac{f(a)+f(b)}{2}(b-a),$$

and the

$$\text{Midpoint Rule,}\quad \int_a^b f(x)\,dx \cong f\left(\frac{a+b}{2}\right)(b-a),$$

for a function f defined on $[a, b]$. The former rule approximates the integral by replacing the graph of f with the line segment from $(a, f(a))$ to $(b, f(b))$, while the latter approximates the integral by replacing the graph of f with the horizontal line segment through $(\frac{a+b}{2}, f(\frac{a+b}{2}))$.

In practice, the interval $[a, b]$ is divided up into a large number of subintervals and an approximation is applied over each subinterval. The hope is that the larger the number of subintervals, the better the approximation becomes. This hope, of course, rests on the nature of f itself—indeed f may be very complicated, or only partially known. This is why estimates for the accuracies of these approximations are important.

For example, if $|f''| \le M$ on $[a, b]$ (at the endpoints, we mean derivatives from the right and from the left) then it is well known that for the

$$\text{Trapezoid Rule,}\quad \left|\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a)\right| \le M\frac{(b-a)^3}{12},$$

and for the

$$\text{Midpoint Rule,}\quad \left|\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a)\right| \le M\frac{(b-a)^3}{24}.$$

Typically these estimates are obtained via polynomial interpolation [**1**], which is by no means elementary. However, D. Cruz-Uribe and C. J. Neugebauer [**2**], obtained the Trapezoid Rule estimate above by a clever application of integration by parts, making it fully accessible to any calculus student.

Our investigation, which may serve as a companion to Cruz-Uribe and Neugebauer's paper, has two parts. First, we modify the idea used there to obtain the Midpoint Rule estimate. Then we extend the idea to obtain Simpson's Rule, thus addressing a problem posed in their paper namely, "it is an open problem to extend our ideas to give an elementary proof of this result."

The Trapezoid Rule Cruz-Uribe and Neugebauer employ what they call integration by parts "backwards." To illustrate what this means, we begin with an integrable function h on $[a, b]$, and we let H be an antiderivative:

$$H(t) = \int_a^t h(x)\,dx.$$

Then for another function f with bounded second derivative, integration by parts (twice) gives

$$\int_a^b f\,dh - hf\Big|_a^b = -\int_a^b h\,df = -\int_a^b f'\,dH = \int_a^b H\,df' - f'H\Big|_a^b$$
$$= \int_a^b H(t) f''(t)\,dt - f'H\Big|_a^b.$$

That is,

$$\int_a^b f dh - hf\Big|_a^b = \int_a^b H(t) f''(t)\,dt - f'H\Big|_a^b. \tag{1}$$

Then in (1) Cruz-Uribe and Neugebauer essentially set

$$h(x) = x - \frac{a+b}{2},$$

in order to "pick up" the values of f at the endpoints a and b. See the graph of h in FIGURE 1 below.

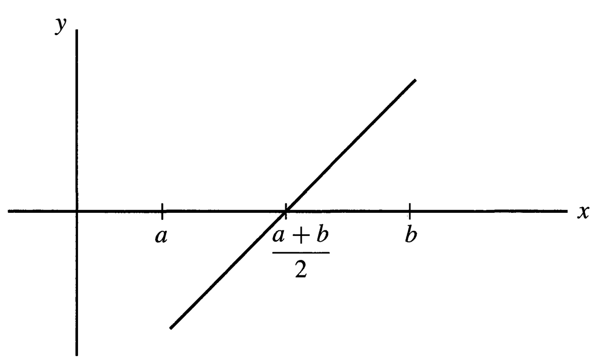

Figure 1 Graph of h for the Trapezoid Rule

Indeed, here $dh = dx$ and $H(a) = H(b) = 0$, so (1) becomes

$$\int_a^b f dh - hf\Big|_a^b = \int_a^b f(x)\,dx - \frac{f(a) + f(b)}{2}(b - a) = \int_a^b H(t) f''(t)\,dt.$$

Now it is easily verified that $\int_a^b |H(t)|\,dt = \frac{(b-a)^3}{12}$, and so we get the Trapezoid Rule estimate

$$\left|\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a)\right| = \left|\int_a^b H(t)f''(t)\,dt\right|$$
$$\leq M\int_a^b |H(t)|\,dt = M\frac{(b-a)^3}{12}.$$

The Midpoint Rule Here instead, to try to "pick up" the values of f at the midpoint $\frac{a+b}{2}$, we'd like to set

$$h(x) = \begin{cases} x-a & \text{if } x \in [a, \frac{a+b}{2}] \\ x-b & \text{if } x \in (\frac{a+b}{2}, b], \end{cases}$$

but strictly speaking (1) does not apply, because h has a jump discontinuity at $(a+b)/2$. (FIGURE 2 shows the graph of h.) So we'll have to be a bit more careful, applying the formula to each half of the interval and taking limits.

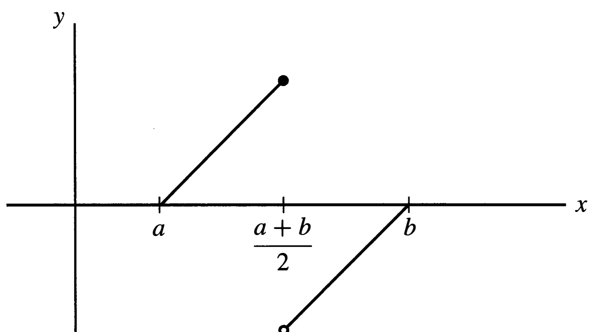

Figure 2 Graph of h for the Midpoint Rule

To allow for the jump at $c := (a+b)/2$, we modify (1) as follows:

$$\int_a^b f(x)\,dt - \lim_{\tau\to c^-} h(x)f(x)\Big|_a^\tau - \lim_{\tau\to c^+} h(x)f(x)\Big|_\tau^b$$
$$= \int_a^b H(t)f''(t)\,dt - f'H\Big|_a^b. \tag{2}$$

Still we have $dh = dx$ and $H(t) = \int_a^t h(x)\,dx$ satisfies $H(a) = H(b) = 0$, so (2) becomes

$$\int_a^b f(x)\,dt - \lim_{\tau\to c^-} h(x)f(x)\Big|_a^\tau - \lim_{\tau\to c^+} h(x)f(x)\Big|_\tau^b$$
$$= \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a) = \int_a^b H(t)f''(t)\,dt.$$

Thus, using $\int_a^b |H(t)|\,dt = \frac{(b-a)^3}{24}$, we obtain the Midpoint Rule estimate

$$\left|\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a)\right| = \left|\int_a^b H(t)f''(t)\,dt\right|$$
$$\le M\int_a^b |H(t)|\,dt = M\frac{(b-a)^3}{24}.$$

More precise estimates As often happens in mathematics, if we ask for more then we get more. If we assume that f has a *continuous* second derivative, we can obtain more precise estimates, as follows. If we look carefully at the graph of the h used to obtain the Trapezoid Rule estimate, we can see that H does not change sign on $[a, b]$—it's nonpositive—and so by the Mean Value Theorem for Integrals [**1**] there exists $\xi_1 \in (a, b)$ such that

$$\int_a^b H(t)f''(t)\,dt = f\ (\xi_1)\int_a^b H(t)\,dt.$$

As one may verify, $\int_a^b H(t)\,dt = -\frac{(b-a)^3}{12}$ and so we have for the

$$\text{Trapezoid Rule,}\qquad \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) = -f''(\xi_1)\frac{(b-a)^3}{12}.$$

Likewise, the H from the Midpoint Rule estimate above does not change sign on $[a, b]$—it's nonnegative—and so there exists $\xi_2 \in (a, b)$ such that

$$\int_a^b H(t)f''(t)\,dt = f\ (\xi_2)\int_a^b H(t)\,dt.$$

Here, $\int_a^b H(t)\,dt = \frac{(b-a)^3}{24}$, and so for the

$$\text{Midpoint Rule,}\qquad \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a) = f''(\xi_2)\frac{(b-a)^3}{24}\ .$$

Simpson's Rule Looking at the more precise error terms for the Trapezoid and Midpoint Rules, we notice that $\frac{1}{3}(-\frac{(b-a)^3}{12}) + \frac{2}{3}\frac{(b-a)^3}{24} = 0$. This suggests (as observed by many authors) that a quadrature rule

$$\frac{1}{3}(\text{Trapezoid Rule}) + \frac{2}{3}(\text{Midpoint Rule})$$

may be quite good. Indeed it is—this is Simpson's Rule.

Pursuing this idea according to what we have done above, we let

$$h_1(x) = x - \frac{a+b}{2} \qquad \text{and} \qquad h_2(x) = \begin{cases} x-a & \text{if } t \in [a, \frac{a+b}{2}] \\ x-b & \text{if } t \in (\frac{a+b}{2}, b]. \end{cases}$$

Now we apply (2) to $h = \frac{1}{3}h_1 + \frac{2}{3}h_2$ and $H(t) = \int_a^t h(x)\,dx$. We omit many of the details; they are elementary but admittedly tedious. The left side of (2) simplifies to become

$$\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

On the right side, one can check that $H(a) = H(b) = 0$, so we apply integration by parts as before to obtain

$$\int_a^b H(t)f''(t)\,dt - f'H\Big|_a^b = \int_a^b H(t)f''(t)\,dt$$
$$= H_1(t)f''(t)\Big|_a^b - \int_a^b H_1(t)f'''(t)\,dt,$$

where $H_1(t) = \int_a^t H(x)\,dx$. Again, $H_1(a) = H_1(b) = 0$, so we apply integration by parts another time to obtain

$$H_1(t)f''(t)\Big|_a^b - \int_a^b H_1(t)f'''(t)\,dt = -\int_a^b H_1(t)f'''(t)\,dt$$
$$= -H_2(t)f'''(t)\Big|_a^b + \int_a^b H_2(t)f^{(iv)}(t)\,dt,$$

where $H_2(t) = \int_a^t H_1(x)\,dx$. Here, yet again, $H_2(a) = H_2(b) = 0$, and so we have thus far

$$\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = \int_a^b H_2(t)f^{(iv)}(t)\,dt.$$

As before, $H_2(t)$ does not change sign on $[a, b]$ and so, for $f^{(iv)}$ continuous, by the Mean Value Theorem for Integrals there exists $\xi \in (a, b)$ such that

$$\int_a^b H_2(t)f^{(iv)}(t)\,dt = f^{(iv)}(\xi)\int_a^b H_2(t)\,dt.$$

Finally, evaluating $\int_a^b H_2(t)\,dt$, we obtain the classical error estimate [1] for

$$\text{Simpson's Rule,}\quad \int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$
$$= -f^{(iv)}(\xi)\frac{(b-a)^5}{2880}.$$

As mentioned, we omitted many of the details. But the intrepid reader may check, for example, that for $[a, b] = [0, 1]$ we have

$$0 \geq H_2(t) = \begin{cases} \frac{1}{72}(3t^4 - 2t^3) & t \in [0, 1/2] \\ \frac{1}{72}(3t^4 - 10t^3 + 12t^2 - 6t + 1) & t \in (1/2, 1]. \end{cases}$$

Simpson's Rule approximates the integral by replacing the graph of f with the parabola through $(a, f(a))$, $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$, and $(b, f(b))$. *Higher order quadrature rules* replace f with higher order polynomials. In principle, the above idea could be extended to these rules, but unless a manageable pattern can be discerned the computations quickly become unwieldy. As such, methods for obtaining estimates for higher order quadrature rules remain, for now, within the realm polynomial interpolation.

Acknowledgment. The first author was supported by a Buffalo State College Research Foundation Undergraduate Summer Research Fellowship. Both authors are grateful to the referees for making suggestions which vastly improved this manuscript.

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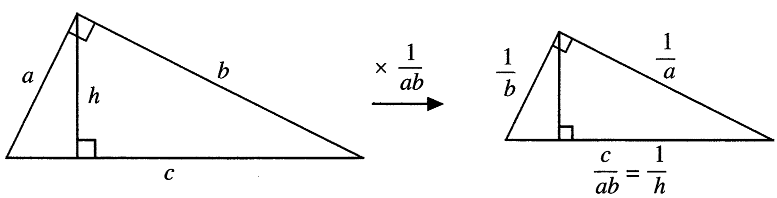
Proof Without Words: A Reciprocal Pythagorean Theorem

doi:10.4169/002557009X478427

If a and b are the legs and h the altitude to the hypotenuse c of a right triangle, then

$$\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 = \left(\frac{1}{h}\right)^2.$$

Proof.



NOTE: For another proof, see V. Ferlini, Mathematics without (many) words, *College Math. J.* **33** (2002) 170.

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Geometry in Government

In a family of accomplished scholars, "my performance was decidedly mediocre. I approached the bulk of my schoolwork as a chore rather than an intellectual adventure."

Geometry came to the rescue. "Instead of memorizing facts, we were asked to think in clear, logical steps. Beginning from a few intuitive postulates, far-reaching consequences could be derived, and I took immediately to the sport of proving theorems."

—Steven Chu
United States Secretary of Energy
as quoted by Lisa M. Krieger
Bay Area News Group

Sines and Cosines of Angles in Arithmetic Progression

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doi:10.4169/002557009X478436

In a recent Math Bite in this MAGAZINE [**2**], Judy Holdener gives a physical argument for the relations

$$\sum_{k=1}^{N} \cos\left(\frac{2\pi k}{N}\right) = 0 \qquad \text{and} \qquad \sum_{k=1}^{N} \sin\left(\frac{2\pi k}{N}\right) = 0,$$

and comments that "It seems that one must enter the realm of complex numbers to prove this result." In fact, these relations follow from more general formulas, which can be proved without using complex numbers. We state these formulas as a theorem.

THEOREM. *If $a, d \in \mathbb{R}$, $d \neq 0$, and n is a positive integer, then*

$$\sum_{k=0}^{n-1} \sin(a + kd) = \frac{\sin(nd/2)}{\sin(d/2)} \sin\left(a + \frac{(n-1)d}{2}\right)$$

and

$$\sum_{k=0}^{n-1} \cos(a + kd) = \frac{\sin(nd/2)}{\sin(d/2)} \cos\left(a + \frac{(n-1)d}{2}\right).$$

We first encountered these formulas, and also the proof given below, in the journal *Arbelos*, edited (and we believe almost entirely written) by Samuel Greitzer. This journal was intended to be read by talented high school students, and was published from 1982 to 1987. It appears to be somewhat difficult to obtain copies of this journal, as only a small fraction of libraries seem to hold them.

Before proving the theorem, we point out that, though interesting in isolation, these formulas are more than mere curiosities. For example, the function $D_m(t) = \frac{1}{2} + \sum_{k=1}^{m} \cos(kt)$ is well known in the study of Fourier series [**3**] as the Dirichlet kernel. This function is used in the proof of Dirichlet's theorem: If a function $f(t)$ is continuous on $[-\pi, \pi]$ and has $f(-\pi) = f(\pi)$, then the Fourier series for $f(t)$ converges to $f(t)$ at every point of $[-\pi, \pi]$. In fact, Pinkus and Zafrany [**3**] prove that

$$D_m(t) = \frac{\sin\left(m + \frac{t}{2}\right)}{2\sin\left(\frac{t}{2}\right)}$$

using the same method as Greitzer.

Proof of the Theorem. We reiterate that this proof is not original with the author. Greitzer [**1**] proves the first formula and leaves the second as an exercise for the reader. Therefore, in this note we will prove the second formula and refer the reader to Grietzer for the other, which proceeds along exactly the same lines. First, recall the trigonometric identity

$$\sin(A+B) - \sin(A-B) = 2\cos(A)\sin(B). \tag{1}$$

Now, if we write C for the sum we wish to evaluate, then multiplying by $2\sin(d/2)$ and using (1) yields

$$\begin{aligned} 2C \cdot \sin(d/2) &= \sum_{k=0}^{n-1} 2\cos(a+kd)\sin(d/2) \\ &= \sum_{k=0}^{n-1} \sin\left(a + \frac{(2k+1)d}{2}\right) - \sin\left(a + \frac{(2k-1)d}{2}\right) \\ &= \sin\left(a + \frac{(2n-1)d}{2}\right) - \sin\left(a - \frac{d}{2}\right) \\ &= 2\cos\left(a + \frac{(n-1)d}{2}\right)\sin(nd/2), \end{aligned}$$

where the third equality follows because the series telescopes. Solving our final equation for C completes the proof of the theorem. ∎

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Black Holes through *The Mirrour*

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doi:10.4169/193009809X468724

What is the path of a free-falling stone when dropped from the surface of a spinning earth? This riddle is an old one. By the late Middle Ages it was common knowledge. In the first printed book with illustrations in the English language, William Caxton in a popular encyclopedia of what everyone should know, called *The Mirrour of the World*, which appeared in 1481 and was a translation of a French manuscript of 1245, which in turn was a translation of an earlier text in Latin, said [**1**, p. 55],

> And if the earth were pierced through in two places, of which that one hole were cut into the other like a cross, and four men stood right at the four heads of these two holes, one above and another beneath, and like-wise on both sides, and that each of them threw a stone into the hole, whether it were great or little, each stone should come into the middle of the earth without ever to be removed from thence.

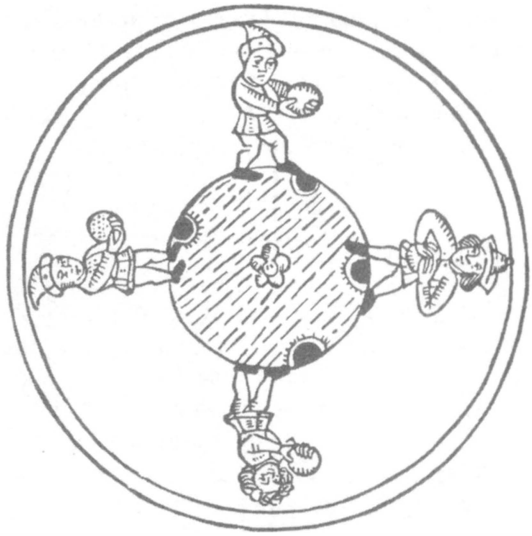

Figure 1 Holes through the earth's center, a Caxton wood-cut of 1481

Caxton illustrates these dynamics with the charming woodcut of FIGURE 1.

Of course, the riddle must remain a mind experiment, because stones cannot fall without resistance through the earth, right? No! As a potential power source, Stephen Hawking proposed dropping a black hole with mass the size of a mountain from the surface of the earth [**4**, pp. 108–109]. Since such a black hole is so tiny, it falls back and forth through the earth without resistance, rarely if ever striking any other particle. How to harness the energy of this motion we leave as a mystery. Instead we focus on the black hole's path through the classic ideal model or mirror of our world, namely, a spinning, homogeneously dense, round earth. If this black hole pebble is dropped at the equator, we will see that its path through the earth is a familiar curve.

Furthermore we generalize the problem, enlisting a supernatural agent to do so. Suppose that the Norse god Thor has the ability both to change the rotation rate of earth at will and to manipulate Hawking's black hole. Varying William Caxton's suggestion in the woodcut of FIGURE 1 about throwing a stone down a hole from the equator, Thor, on a stationary earth, throws the black hole at a tangent to the equator. Provided Thor's toss is not overly fast, the black hole will yo-yo through the earth. Meanwhile, Thor sets the earth to spinning. This change in the motion of the earth has no effect upon the motion of the black hole in any way, but it does change the path of the hole through the earth, generating a family of classic curves that are presented in the parametric equations section of almost any calculus text. However, if such a pebble is dropped from northern climes, the corresponding hole fails to be a planar curve, let alone a member of this classic family. A little linear algebra is enough to demonstrate these results.

A little history The second century writer Plutarch suggested that a boulder falling through a spherical earth would pass through the earth's center and proceed to the other side, and then fall back, retracing its path, and fall again, and so on forever, somewhat like the motion of a balance scale or a swing [**6**, p. 65]. Galileo, in his great dialogue about motion, used a cannonball in his thought experiment, and improved Plutarch's description of the back and forth motion of the ball, so quantifying simple harmonic motion [**3**, p. 227]. About fifty years later, Isaac Newton began the *Principia* with a discussion of Galileo's problem. He imagined a pebble falling without resistance through a homogeneously dense earth. With respect to the background of the fixed stars, he showed that the path of the pebble is an ellipse whose center is earth's center in Corollary i of Proposition 10 of Book I of the *Principia* [**5**, p. 460].

In particular, if the pebble is dropped at earth's equator, the path of the pebble in the plane of the equator is parametrized, in x-y coordinates, by

$$P(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = R \begin{bmatrix} \cos(\sqrt{k}\,t) \\ \frac{\omega}{\sqrt{k}} \sin(\sqrt{k}\,t) \end{bmatrix}, \tag{1}$$

where R is earth's radius, ω is earth's angular frequency about its axis, t is time, and $\sqrt{k}$ is a constant relating the gravitational acceleration f on the pebble at distance r from earth's center: $f(r) = -kr$. The pebble falls with simple harmonic motion in both its x and y coordinates. Although Newton demonstrates these results using geometry, a straight-forward derivation of this result using conventional notation has appeared in the MAGAZINE [**7**]. For our earth, $R \approx 6400$ km, $\omega = 2\pi$ radians per day, $kR \approx 9.8$ m/sec^2, the graph of (1) has semi-minor axial length of about 276 km, and is depicted in FIGURE 2(a).

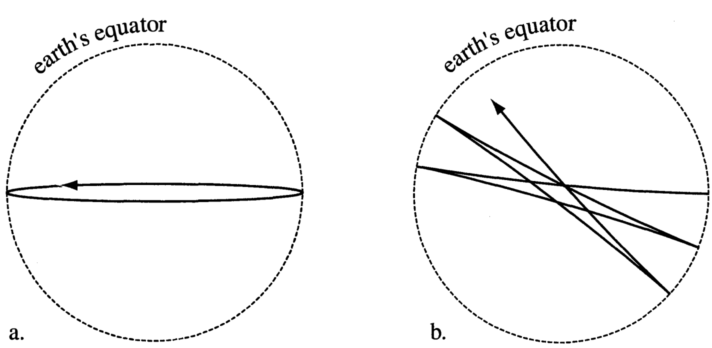

Figure 2 Pebble motion with respect to the stars and to the earth

While (1) is an ellipse with respect to the fixed stars, what is its path through the rotating earth? To find it, we simply rotate (1) in a special way. For ease of notation, for $|b| < 1$, we say that

$$\mathcal{E}(b, t) = \begin{bmatrix} \cos t \\ b \sin t \end{bmatrix} \tag{2}$$

is a *normalized* parametrization in t of an ellipse because, with respect to t, it parametrizes an ellipse with semi-major axial length 1 and semi-minor axial length b. Thus (1) is $R\,\mathcal{E}(\omega/\sqrt{k}, \sqrt{k}\,t)$. (Equivalently, (2) is (1) when distance and time units are defined so that $R = 1$ and $\sqrt{k} = 1$ with $|b| = \omega$.) To rotate a parametrization, we use a *dynamic* rotation matrix, so-called because the angle of rotation changes with respect to time. Let $\mathcal{Q}(\alpha, t)$ be the 2×2 matrix that when multiplied to a two-dimensional vector will rotate it αt radians counterclockwise about the origin O,

$$\mathcal{Q}(\alpha, t) = \begin{bmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{bmatrix}.$$

Since the earth rotates ωt radians counterclockwise in a time lapse of t, then with respect to the earth, the falling pebble is at $\mathcal{Q}(-\omega, t)P(t)$; that is, at time t, the pebble must be rotated ωt radians clockwise from its position with respect to the stars so as to obtain its position with respect to the earth. The graph of this hole through the earth parametrized in time by $\mathcal{Q}(-\omega,\ t)P(t)$ is shown in FIGURE 2(b).

We define $\mathcal{H}(\alpha,\ b,\ t)$ as

$$\mathcal{Q}(-\alpha,\ t)\mathcal{E}(b,\ t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos\alpha t \cos t + b \sin\alpha t \sin t \\ -\sin\alpha t \cos t + b \cos\alpha t \sin t \end{bmatrix}, \tag{3}$$

a parametric *precession* of $\mathcal{E}(b,\ t)$ in t by the angular frequency α, where $|b| < 1$, and that its graph is a *precession* of the ellipse parametrized by $\mathcal{E}(b,\ t)$.

To imagine the dynamics of (3), we let Thor throw a black hole along a tangent to the equator while standing on a stationary earth. The black hole follows an ellipse in the equatorial plane given by $\mathcal{E}(b,\ t)$ for some b. Next, Thor imparts to the earth an angular frequency of α. While this motion change has no effect upon the black hole, its position with respect to the rotating earth at time t is $\mathcal{H}(\alpha,\ b,\ t)$. FIGURE 3 shows the graphs of some of these holes. They look suspiciously like the graphs of what the typical calculus text classifies as *trochoids*.

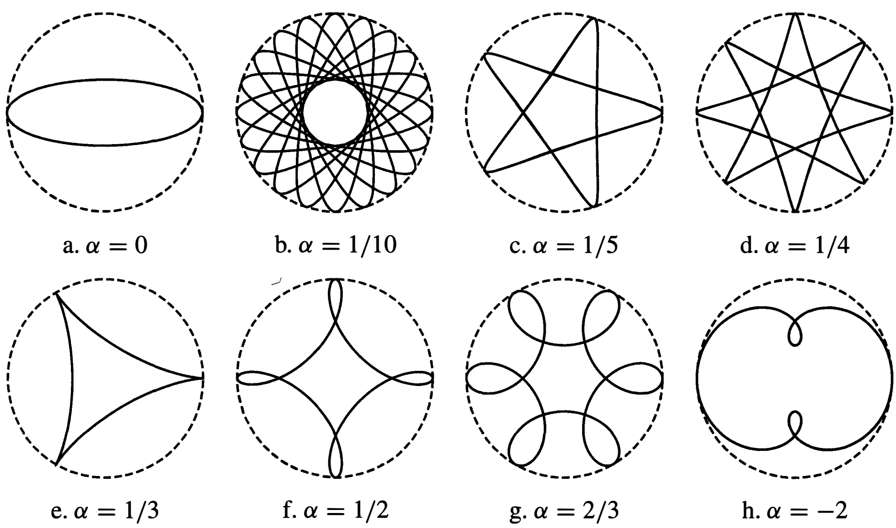

Figure 3 Trochoidal-like paths through the earth, $b = 1/3$

Trochoid parametrizations Informally, a *trochoid* is the curve traced by a tack stuck in a wheel as the wheel rolls (without slipping) along either the inside or the outside of a circle. When the tack is on the rim of the wheel rolling along the outside of a circle, the tack traces an *epicycloid.* When the tack is offset from the rim, the generated curve is an *epitrochoid*, a curve that the ancients used to model the paths of planets in a geocentric universe. When the tack is on the wheel rolling along the inside of a circle, the tack traces a *hypocycloid.* When such a tack is offset from the rim, the generated curve is called a *hypotrochoid.*

We consider the hypotrochoid first. Suppose a wheel D is a disk of radius d that rolls along the inside of a circle C of radius c, where $c > d > 0$. Let T be a tack r units out along a ray from a wheel's hub through a point on its rim, with $r > 0$. Position the origin O at C's center in the xy-plane. Interpret a parameter θ as the central angle at O between the positive x-axis and D's center, $D(\theta)$. In terms of θ, $D(\theta) = (c - d)(\cos\theta,\ \sin\theta)$. At $\theta = 0$, let $D(0) = (c - d,\ 0)$, with T at the point $(c - d + r,\ 0)$. Let $\phi(\theta)$ be the angle between the direction $\mathbf{i}$ and the tack, where $\mathbf{i}$ is the unit vector $(1, 0)$, as illustrated in FIGURE 4, which shows the hypotrochoid for the case $3d = c$ with $r = 1.25d$. As shown in almost any calculus text [**2**, pp. 689], the standard parametrization of the hypotrochoid is

$$(c-d)\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} + r\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}, \tag{4}$$

where $\phi = (d-c)\theta/d$.

Now let $\lambda = (c-d)/(c-d+r)$, $\mu = r/(c-d+r) = 1-\lambda$, $p = d$, $q = c-d$, and let t be a new parameter with $\theta = pt$. An equivalent form to (4) is

$$(c-d+r)\left(\lambda\begin{bmatrix}\cos pt\\ \sin pt\end{bmatrix} + \mu\begin{bmatrix}\cos(-qt)\\ \sin(-qt)\end{bmatrix}\right). \tag{5}$$

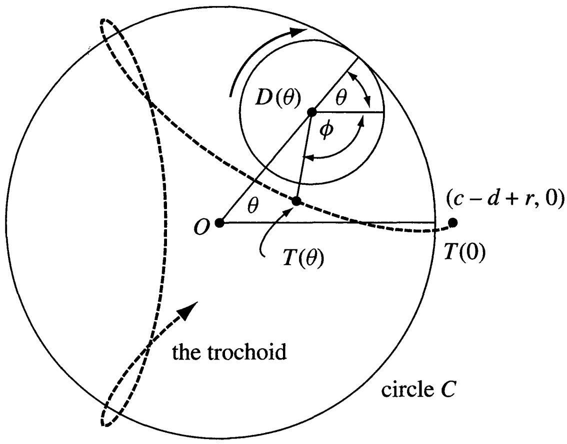

Figure 4 Generating a hypotrochoid

That is, given valid values for c, d, and r, (4) can be transformed into (5), and given λ, μ, p, and q with $0 < \lambda < 1$, $\mu = 1-\lambda$, $p > 0$, $q > 0$, then $d = p$, $c = q + p$, $r = \mu(c-d)/\lambda$, and $\theta = pt$, which means that (5) can be transformed into (4). Hence, we say that a *normalized* parametrization of the hypotrochoid is

$$h(p, q, \lambda, \mu, t) = \lambda\begin{bmatrix}\cos pt\\ \sin pt\end{bmatrix} + \mu\begin{bmatrix}\cos(-qt)\\ \sin(-qt)\end{bmatrix}, \tag{6}$$

where $0 < \lambda < 1$, $\mu = 1-\lambda$, and p and q have the same sign. The beauty of (6) over (4) is that its graph is inscribed in the unit circle for all valid values of λ, μ, p, and q. Thus any hypotrochoid is a scaled version of (6). In particular, when $\lambda = q/(p+q)$ (which corresponds to $r = d$), then

$$h(p, q, q/(p+q), p/(p+q), t) \tag{7}$$

is a hypocycloid.

Similarly, the standard parametrization of an epitrochoid is

$$(c+d)\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} + r\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}, \tag{8}$$

where $\phi = (c+d)\theta/d$, with $c > 0, d > 0$, and $r > 0$. With $\lambda = (c+d)/(c+d+r)$, $\mu = r/(c+d+r) = 1-\lambda$, $p = d$, $q = c+d$, and $\theta = dt$, an equivalent form to (8) is

$$(c+d+r)\left(\lambda\begin{bmatrix}\cos pt\\ \sin pt\end{bmatrix}+\mu\begin{bmatrix}\cos qt\\ \sin qt\end{bmatrix}\right). \tag{9}$$

Comparing (5) and (9), we can collapse the hypotrochoids and epitrochoids into a single normalized form,

$$\mathcal{T}(p,\, q,\, \lambda,\, \mu,\, t) = \lambda\begin{bmatrix}\cos pt\\ \sin pt\end{bmatrix}+\mu\begin{bmatrix}\cos qt\\ \sin qt\end{bmatrix}, \tag{10}$$

where $0 < \lambda < 1$, $\mu = 1-\lambda$, and p and q are nonzero. If p and q have the same sign then $\mathcal{T}(p,\, q,\, \lambda,\, \mu,\, t)$ is an epitrochoid, and if p and q have opposite signs then $\mathcal{T}(p,\, q,\, \lambda,\, \mu,\, t)$ is a hypotrochoid. In particular, by (7), for any real number b with $|b| < 1$,

$$\mathcal{T}\left(1-b,\, -(1+b),\, \frac{1+b}{2},\, \frac{1-b}{2},\, t\right) \tag{11}$$

parametrizes a hypocycloid.

The trochoid as a precessed ellipse To show that (3) parametrizes trochoids, we write the normalized ellipse in trochoid form as

$$\begin{aligned}\mathcal{E}(b,\, t) = \begin{bmatrix}\cos t\\ b\sin t\end{bmatrix} &= \frac{1+b}{2}\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}+\frac{1-b}{2}\begin{bmatrix}\cos(-t)\\ \sin(-t)\end{bmatrix}\\ &= \mathcal{T}\left(1,\, -1,\, \frac{1+b}{2},\, \frac{1-b}{2},\, t\right).\end{aligned}$$

By the addition identities for cosine and sine,

$$\mathcal{Q}(\alpha,\, t)\begin{bmatrix}\cos \beta t\\ \sin \beta t\end{bmatrix} = \begin{bmatrix}\cos(\alpha+\beta)t\\ \sin(\alpha+\beta)t\end{bmatrix} = \mathcal{Q}(\alpha+\beta,\, t)\begin{bmatrix}1\\ 0\end{bmatrix}.$$

Therefore,

$$\begin{aligned}\mathcal{H}(\alpha,\, b,\, t) &= \mathcal{Q}(-\alpha,\, t)\mathcal{E}(b,\, t) = \mathcal{Q}(-\alpha,\, t)\begin{bmatrix}\cos t\\ b\sin t\end{bmatrix}\\ &= \frac{1+b}{2}\mathcal{Q}(-\alpha,\, t)\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}+\frac{1-b}{2}\mathcal{Q}(-\alpha,\, t)\begin{bmatrix}\cos(-t)\\ \sin(-t)\end{bmatrix}\\ &= \mathcal{T}\left(-\alpha+1,\, -\alpha-1,\, \frac{1+b}{2},\, \frac{1-b}{2},\, t\right),\end{aligned}$$

which means that $\mathcal{H}(\alpha,\, b,\, t)$ parametrizes a trochoid. To reverse the argument, consider a typical trochoid E whose graph is given by $\mathcal{T}(p,\, q,\, \lambda,\, \mu,\, t)$. To reverse the argument, consider a trochoid E parametrized by $\mathcal{T}(p,\, q,\, \lambda,\, \mu,\, t)$. Solving $\lambda = (1+b)/2$ and $\mu = (1-b)/2$ gives $b = \lambda - \mu$, which means that $|b| < 1$. Solving $p = 1-\alpha$ and $q = -(1+\alpha)$ gives $\alpha = -(p+q)/2$. So E is parametrized by $\mathcal{H}(-(p+q)/2,\, \lambda-\mu,\, t)$. That is, $\mathcal{T}(p,\, q,\, \lambda,\, \mu,\, t)$ and $\mathcal{H}(-(p+q)/2,\, \lambda-\mu,\, t)$

are the same parametrization. Therefore, any trochoid is also a precessed ellipse and can be, courtesy of Thor, perceived as the hole followed by a black hole through the earth. Furthermore, when $\alpha = b$, then $\mathcal{H}(b,\ b,\ t) = \mathcal{T}(1-b,\ -(1+b),\ (1+b)/2,\ (1-b)/2,\ t)$; by (11)

$$\mathcal{H}(b,\ b,\ t) \tag{12}$$

parametrizes a hypocycloid.

Finally, we show that $\mathcal{Q}(-\omega,\ t)P(t)$ is a hypocycloid. Since $P(t) = R\,\mathcal{E}(\omega/\sqrt{k},\ \sqrt{k}\,t)$, then

$$\mathcal{Q}(-\omega,\ t)P(t) = R\,\mathcal{Q}(-\omega,\ t)\mathcal{E}\left(\frac{\omega}{\sqrt{k}},\ \sqrt{k}\,t\right) = R\,\mathcal{H}\left(\frac{\omega}{\sqrt{k}},\ \frac{\omega}{\sqrt{k}},\ \sqrt{k}\,t\right),$$

which traces the same curve as $R\,\mathcal{H}(\omega/\sqrt{k},\ \omega/\sqrt{k},\ t)$, a hypocycloid by (12) when $\omega/\sqrt{k} < 1$. (For our earth $\omega/\sqrt{k} \approx 0.059$, a unitless quantity.) That is, when a pebble is dropped from the equator of a rotating, homogeneous earth, its path through the earth is a hypocycloid.

Holes starting north of the equator What kind of hole results when the pebble is dropped from northern latitudes on a rotating earth?

In particular, we drop a pebble at latitude ψ. As before, with respect to the fixed stars, the pebble's path is an ellipse. Since the pebble's initial tangential speed at latitude ψ is less by a factor of $\cos\psi$ in comparison to the initial tangential speed of a particle at the earth's equator, then the semi-minor axial length for the ellipse originating at latitude ψ will be $\cos\psi$ of the semi-minor axial length for the ellipse originating at the equator. That is, in comparison to (1), a parametrization for the ellipse originating at latitude ψ is

$$R\begin{bmatrix} \cos(\sqrt{k}\,t) \\ \frac{\omega}{\sqrt{k}}\cos\psi\sin(\sqrt{k}\,t) \end{bmatrix}.$$

To this parametrization, append a third component with value 0, and call the new parametrization $P(t,\ \psi)$:

$$P(t,\ \psi) = R\begin{bmatrix} \cos(\sqrt{k}\,t) \\ \frac{\omega}{\sqrt{k}}\cos\psi\sin(\sqrt{k}\,t) \\ 0 \end{bmatrix}.$$

Let $L(\psi)$ be the matrix that, when multiplied to a vector, rotates it about the y-axis by angle ψ, counterclockwise with respect to the positive y direction:

$$L(\psi) = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix}.$$

Thus, $L(\psi)P(t,\ \psi)$ rotates the ellipse into alignment with the trajectory's natural position with respect to the equatorial plane being the xy-plane and the drop point (at $t = 0$) being at latitude ψ. Let $M(t)$ be the matrix that, when multiplied to a vector, rotates it about the z-axis by ωt radians, counterclockwise with respect to the positive z direction:

$$M(t) = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, $\mathcal{F}(t, \psi) = M(-t)L(\psi)P(t, \psi)$ is a parametrization of the pebble's position with respect to the rotating earth. The graph of $\mathcal{F}(t, \psi)$ for a given ψ is the hole through the earth drilled by a pebble dropped at latitude ψ for an earth rotating with angular frequency ω and radius R.

As an example, we drop a pebble at Torino, Italy, 45° N, 7.5° E, which we call point A. At what longitude will the pebble resurface? With ω as one rotation per day, $\mathcal{F}(\pi/\sqrt{k}, \pi/4)$ gives $(-4448.6, 830.8, -4525.5)$, which means that the resurface point is about 169.4° east of Torino, giving the resurface point as 45° S and 177° E, which is about 300 miles southeast off the coast from Wellington, New Zealand. However, if the earth rotates at, say, once every two hours (and the earth maintains its spherical shape), then the resurface point B is at 45° S and 60.5° E, putting it well off the southeast coast of Africa, out where the Indian Ocean merges with Antarctic waters. FIGURE 5 shows the geometry of these holes.

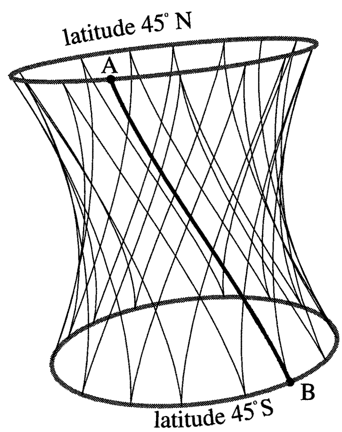

Figure 5 A wicker basket, $\omega = \pi$ radians/hr

The holes are nonplanar, even from A to B. We demonstrate this result for the case when dropping a black hole at latitude $\psi = \pi/4$. Without loss of generality, we may define a time unit so that $1/\sqrt{k}$ is unit time, and a distance unit so that R is unit distance. Under such a scheme, the valid values for ω are between 0 and 1 radian per unit time. (For any higher value of ω, a black hole dropped at the equator would fly off into space.) Let $A = \mathcal{F}(0, \psi)$, $B = \mathcal{F}(\pi, \psi)$, $C = \mathcal{F}(\pi/2, \psi)$, and $D = \mathcal{F}(\pi/4, \psi)$. Let U be the cross product of $C - A$ and $B - A$, let V be the cross product of $D - A$ and $B - A$, and let ϵ be the magnitude of the difference between the unit vectors in the U and V directions. If the curve given by $\mathcal{F}(t, \psi)$ is planar then U and V point in the same direction, which means that ϵ should be close to 0. However the graph given in FIGURE 6 shows that ϵ is nowhere near 0 as ω goes from 0 to 1 (except at $\omega = 0$). Thus the curve from A to B is nonplanar for any ω with $0 < \omega < 1$.

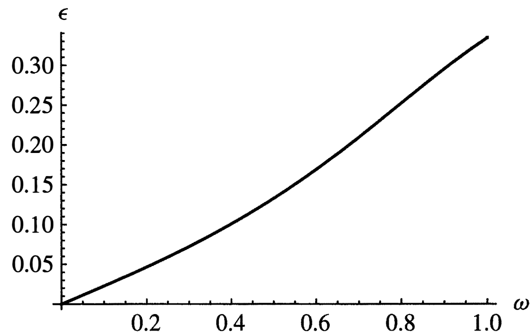

Figure 6 The nonplanarity of nonequatorial tunnels

While the holes given by $\mathcal{F}(t,\ \psi)$ form a network resembling a wicker basket or a vase of reeds, their projection onto the xy-plane is a hypocycloid. To see this result, the first two coordinates of $\mathcal{F}(t,\ \psi)$ are

$$R\cos\psi\begin{bmatrix}\cos(\omega t)\cos(\sqrt{k}\,t)+\frac{\omega}{\sqrt{k}}\sin(\omega t)\sin(\sqrt{k}\,t)\\ -\sin(\omega t)\cos(\sqrt{k}\,t)+\frac{\omega}{\sqrt{k}}\cos(\omega t)\sin(\sqrt{k}\,t)\end{bmatrix}. \tag{13}$$

With $\tau=\sqrt{k}\,t$, (13) becomes a scaling of $\mathcal{H}(\omega/\sqrt{k},\ \omega/\sqrt{k},\ \tau)$, a hypocycloid by (12).

A parting woodcut In closing, consider the wood-cut of FIGURE 7 illustrating a passage from *The Mirrour of the World* in which Caxton explains why the world is round.

> God formed the world all round; for of all the forms that be, of what diverse manners they be, none are so full or receptive so much by nature as is the figure round that may hold within it so much in right quantity as does the round. [**1**, pp. 58–59]

Caxton goes on to show that curves with corners, when rotating, "take diverse places that the round seeks not." Inside the double circle, which represents the earth's form, are two cornered curves, one is a square and the other appears to be a hypocycloid.

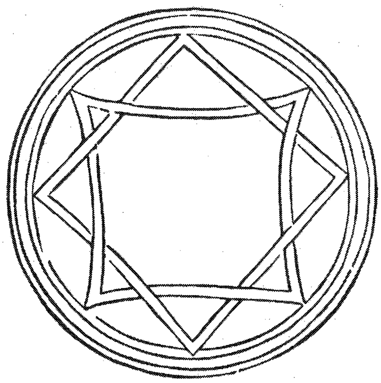

Figure 7 A Caxton hypocycloid?

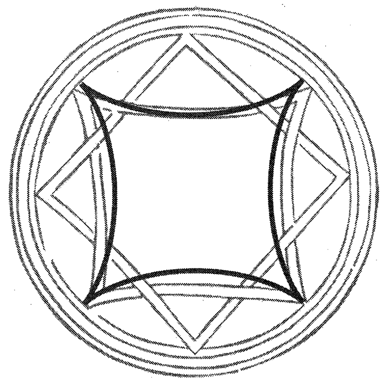

Figure 8 A hypocycloid with four cusps superimposed on a woodcut

In particular, the second curve looks like the hypocycloidal hole followed by a black hole dropped at the equator of an earth rotating once every 2.82 hours. Could it be that Caxton presciently knew about hypocycloids thirty five years before Albrecht Dürer purportedly drew the first one, and knew about resistance-free black hole pebbles five hundred years before Hawking suggested their existence? How close is the match?

Superimposing a graph of a periodic hypocycloid with four cusps onto Caxton's curve gives FIGURE 8. No, the match fails. But for the upper-most arc, the match is uncannily close.

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6. Plutarch, *Plutarch's Moralia*, vol.xii, translated by H. Cherniss and W. C. Helmbold, Harvard University Press, 1957.
7. A. J. Simoson, Falling down a hole through the earth, this MAGAZINE, **77**:3 (2004) 171–189.

Poem: The Scholar's Song

I am the very model of the scholar mathematical!
I've information plenty of the abstract and the practical!
I know the classic theorems and I note the greats historical,
From Archytas to Andrew Wiles, in order categorical!

I'm very well acquainted too with functional analysis.
I've filled so many notebooks I have median nerve paralysis!
About Fermat's Last Theorem I can come to you with news a lot,
And many cheerful facts about the set named after Mandelbrot!

Chorus: And many cheerful facts about the set named after Mandelbrot!
And many cheerful facts about the set named after Mandelbrot!
And cheerful, cheerful, facts about the set named after Mandel-Mandelbrot!

I'm very good at logic and at all the arts of reasoning,
Although my teaching skills, they say, could use a little seasoning!
But still, with all my knowledge of the abstract and the practical,
I am the very model of the scholar mathematical!

(with apologies to W. S. Gilbert)

—Richard Cleveland
Professor of Mathematics Emeritus
California State University, Sacramento
Sacramento, CA 95819

PROBLEMS

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doi:10.4169/002557009X478445

PROPOSALS

To be considered for publication, solutions should be received by May 1, 2010.

1831. *Proposed by Emeric Deutsch, Polytechnic Institute of NYU, Brooklyn, NY.*

Let k and n be positive integers with $1 \leq k \leq n$, and let $a(n, k)$ be the number of permutations of the set $\{1, 2, \dots, n\}$ for which k is the largest element in the cycle containing 1. Find a closed form expression for $a(n, k)$.

1832. *Proposed by Michel Bataille, Rouen, France.*

Find all solutions to the following system of equations:

$$4x^2 + 8y^2 + 2z^2 + 18xy + 8yz + 9zx = 49(x + 1)$$
$$2x^2 + 4y^2 + 8z^2 + 9xy + 18yz + 8zx = 49(y + 1)$$
$$8x^2 + 2y^2 + 4z^2 + 8xy + 9yz + 18zx = 49(z + 1).$$

1833. *Proposed by Sam Vandervelde and Richard Torres, St. Lawrence University, Canton, NY.*

For prime p, let M_p be the $p^2 \times p^2$ matrix whose ijth entry, $0 \leq i, j \leq p^2 - 1$, is given by

$$m_{i,j} = (i \bmod p)^{(j \bmod p)} \lfloor i/p \rfloor^{\lfloor j/p \rfloor},$$

where $(k \bmod p)$ denotes the remainder when k is divided by p and we take $0^0 = 1$. Prove that

$$\det(M_p) \equiv (-1)^{(p+1)/2} \pmod{p}.$$

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE. Each solution should begin on a separate sheet.

Solutions and new proposals should be mailed to Bernardo M. Abrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St., Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX file) to mathmagproblems@csun.edu. All communications should include the reader's name, full address, and an e-mail address and/or FAX number.

1834. *Proposed by Cosmin Pohoata, student, National College "Tudor Vianu," Bucharest, Romania.*

Let $ABCD$ be a quadrilateral that has an inscribed circle with center I, and let ℓ be a line tangent to the incircle. Let A', B', C' and D', respectively, be the projections of A, B, C, and D onto ℓ. Prove that

$$\frac{AA' \cdot BB'}{CC' \cdot DD'} = \frac{AI \cdot BI}{CI \cdot DI}.$$

1835. *Proposed by Finbarr Holland, University College Cork, Ireland.*

Let T be the so-called *tree function* defined by the power series

$$T(z) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} z^n.$$

For $0 \leq x < \infty$, let $g(x) = T(xe^{-x})$. Show that g is continuous on $[0, \infty)$ and that if $0 < a < 1$, then

$$(1-a)\int_0^{\infty} \left(\frac{g(x)}{x}\right)^a dx + a\int_0^{\infty} \left(\frac{g(x)}{x}\right)^{1-a} dx = a(1-a)\pi^2 \csc^2(\pi a).$$

Quickies

Answers to the Quickies are on page 389.

Q995. *Proposed by Michael Goldenberg and Mark Kaplan, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore, MD.*

Let AD be a median of triangle ABC and let the cevian CE meet AD at F. Suppose

$$\frac{AE}{EB} = \frac{CF}{FE} = \lambda.$$

Find the value of λ.

Q996. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

It is well known that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f(x+y) = f(x) + f(y)$ for all x and y in $\mathbb{R}$, then $f(x) = mx$ for some real constant m.

a. Is the same true if $\mathbb{R}$ is replaced by $\mathbb{C}$? That is, if $f : \mathbb{C} \to \mathbb{C}$ is continuous and satisfies

$$f(w+z) = f(w) + f(z) \quad \text{for all} \quad w, z \in \mathbb{C},$$

then must it be the case that

$$f(z) = az \quad \text{for some complex constant } a?$$

b. If the answer to part a. is no, then does the answer become yes if f is differentiable at 0?

Solutions

An ellipsoid and a plane **December 2008**

1806. *Proposed by Michael Becker, University of South Carolina at Sumter, Sumter, SC.*

The intersection of the ellipsoid $x^2 + y^2 + \frac{z^2}{c^2} = 1$ and the plane $x + y + cz = 0$ is an ellipse. For $c > 1$, find the value of c for which the area of the ellipse is maximal.

Solution by Ken Yanosko, Humboldt State University, Arcata, CA.

Because $c > 1$ and the given plane passes through the center of the ellipsoid, the endpoints of the minor axis of the ellipse are the points of intersection of the plane and the "equator" of the ellipse. These endpoints, found by solving the simultaneous equations $x + y = 0$ and $x^2 + y^2 = 1$, are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$ and $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$. Hence, the minor axis has length 2 and the corresponding semi-axis has length 1. Because the ellipse is symmetric about the plane $x = y$ and this plane does not contain the minor axis, this plane must contain the major axis. The endpoints $\pm(\alpha, \beta, \delta)$ of this axis are solutions to the system of simultaneous equations $x^2 + y^2 + \frac{z^2}{c^2} = 1$, $x + y + cz = 0$ and $x = y$. Solving we find $\delta^2 = \frac{2c^2}{c^4+2}$, and the corresponding semi-axis has length

$$\sqrt{\alpha^2 + \beta^2 + \delta^2} = \sqrt{\left(1 - \frac{\delta^2}{c^2}\right) + \delta^2} = \sqrt{\frac{c^4 + 2c^2}{c^4 + 2}}.$$

This length (and the area of the ellipse) are maximized for $c > 1$ when $c = \sqrt{1 + \sqrt{3}}$ for a maximal cross-sectional area of $\pi\sqrt{\frac{1+\sqrt{3}}{2}}$.

Also solved by George Apostolopoulos (Greece), Armstrong Problem Solvers, Michel Bataille (France), J. C. Binz (Switzerland), Robert Calcaterra, Chip Curtis, Matthew Davis and Xiaoshen Wang, Paul Deiermann, M. J. Englefield, John Ferdinands, Fisher Problem Solving Group, Natacha Fontes-Merz, Leon Gerber, Marshall Gray, Eugene A. Herman, Jonathan P. Hexter, Parviz Khalili, Stavros Kontogiorgis (Cyprus), Omran Kouba (Syria), Victor Y. Kutsenok, Lafayette College Problem Group, Elias Lampakis (Greece), Jeremy Lee, Tim McDevitt, Kim McInturff, Laura Neaville, José Pacheco and Ángel Plaza (Spain), Ossama A. Saleh and Terry J. Walters and Stan Byrd, Joel Schlossberg, Seton Hall University Problem Solving Group, John Sumner and Aida Kadic-Galeb, Nora Thornber, Daniel Narrias Villar (Chile), Michael Vowe (Switzerland), Xiaoshen Wang, and the proposer. There were five incorrect submissions.

A positive zero **December 2008**

1807. *Proposed by Lenny Jones, Shippensburg University, Shippensburg, PA.*

Let P be a polynomial with integer coefficients and let s be an integer such that for some positive integer n, $s^{n+1}P(s)^n$ is a positive zero of P. Prove that $P(2) = 0$.

Solution by Armstrong Problem Solvers, Armstrong Atlantic State University, Savannah, GA.

Because $s^{n+1}P(s)^n$ is a zero of P, we have

$$P(x) = \left(x - s^{n+1}P(s)^n\right) Q(x),$$

where $Q(x)$ is a polynomial with integer coefficients. Thus,

$$P(s) = \left(s - s^{n+1}P(s)^n\right) Q(s). \tag{1}$$

Let $m = sP(s)$. It then follows from (1) that

$$m = s^2(1 - m^n)Q(s),$$

where $Q(s)$ is an integer. Notice that $m \neq 1$, since otherwise this equation would lead to $1 = 0$.

Because $1 = m^{n-1}m + (1 - m^n)$, we conclude that m and $1 - m^n$ are relatively prime. However

$$\frac{m}{1 - m^n} = s^2 Q(s)$$

is an integer, so $1 - m^n = \pm 1$, and $m^n = 0$ or 2. If $m = 0$, then $sP(s) = 0$, so either $s = 0$ or $P(s) = 0$, contradicting the assumption that $s^{n+1}P(s)^n > 0$. Thus, $m^n = 2$, and it follows that $m = sP(s) = 2$ and $n = 1$. Since $s^2 P(s) > 0$, either $s = 2$ and $P(s) = 1$, or $s = 1$ and $P(s) = 2$.

If $s = 2$, then $m = s^2(1 - m^n)Q(s)$, implying that $2 = 4(-1)Q(2)$. This is impossible because $Q(2)$ is an integer. Thus, $s = 1$, $P(s) = 2$, and $s^{n+1}P(s)^n = 2$ is a zero of P.

Also solved by Fisher Problem Solving Group, Nicholas C. Singer, John Sumner and Aida Kadic-Galeb, Bob Tomper, Giang Tran, and the proposer.

An equality of sums **December 2008**

1808. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Let α and β be positive real numbers with $\alpha\beta = \pi$, and let y be a real number. Prove that

$$\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\alpha k}\cos(\alpha y k) = \frac{1}{\alpha}\sum_{j=-\infty}^{\infty} \frac{1}{1 + (y + 2\beta j)^2}.$$

Solution by Michel Bataille, Rouen, France.

It is well known that

$$\cot z = \sum_{j=-\infty}^{\infty} \frac{1}{z + j\pi}$$

for $z \in \mathbb{C}\backslash(\pi\mathbb{Z})$. It follows that for $z \in \mathbb{C}\backslash(i\pi\mathbb{Z})$,

$$i\cot(iz) = \sum_{j=-\infty}^{\infty} \frac{1}{z + ij\pi}.$$

Next note that if $z = \frac{\alpha(1+iy)}{2}$ and $\alpha\beta = \pi$, then for $j \in \mathbb{Z}$ we have

$$\mathrm{Re}\left(\frac{1}{z + ij\pi}\right) = \frac{2}{\alpha}\cdot\frac{1}{1 + (y + 2\beta j)^2}.$$

We deduce that

$$\frac{1}{\alpha}\sum_{j=-\infty}^{\infty} \frac{1}{1 + (y + 2\beta j)^2} = \mathrm{Re}\left(\frac{i}{2}\cot\left(\frac{i\alpha(1 + iy)}{2}\right)\right). \tag{1}$$

On the other hand, for positive integer k,

$$e^{-\alpha k}\cos(\alpha y k) = \mathrm{Re}\left(e^{-\alpha(1+iy)k}\right) = \mathrm{Re}\left((e^{-\alpha(1+iy)})^k\right),$$

and, since $\left|e^{-\alpha(1+iy)}\right| = e^{-\alpha} < 1$,

$$\sum_{k=1}^{\infty}(e^{-\alpha(1+iy)})^k = e^{-\alpha(1+iy)}\cdot\frac{1}{1-e^{-\alpha(1+iy)}} = \frac{e^{-\frac{\alpha(1+iy)}{2}}}{e^{\frac{\alpha(1+iy)}{2}}-e^{-\frac{\alpha(1+iy)}{2}}}.$$

It follows that

$$\frac{1}{2}+\sum_{k=1}^{\infty}(e^{-\alpha(1+iy)})^k = \frac{1}{2}\cdot\frac{e^{\frac{\alpha(1+iy)}{2}}+e^{-\frac{\alpha(1+iy)}{2}}}{e^{\frac{\alpha(1+iy)}{2}}-e^{-\frac{\alpha(1+iy)}{2}}} = \frac{i}{2}\cot\left(\frac{i\alpha(1+iy)}{2}\right).$$

Thus, recalling (1),

$$\begin{aligned}\frac{1}{2}+\sum_{k=1}^{\infty}e^{-\alpha k}\cos(\alpha yk) &= \operatorname{Re}\left(\frac{1}{2}+\sum_{k=1}^{\infty}(e^{-\alpha(1+iy)})^k\right)\\ &= \operatorname{Re}\left(\frac{i}{2}\cot\left(\frac{i\alpha(1+iy)}{2}\right)\right) = \frac{1}{\alpha}\sum_{j=-\infty}^{\infty}\frac{1}{1+(y+2\beta j)^2}.\end{aligned}$$

Also solved by Khristo Boyadzheiv, Hongwei Chen, Chryso Christodoulidou (Cyprus), Paul Deiermann, Eugene A. Herman, Elias Lampakis (Greece), John H. Smith, John Sumner and Aida Kadic-Galeb, Marian Tetiva (Romania), Michael Vowe (Switzerland), and the proposer.

An invariant ratio **December 2008**

1809. *Proposed by Cosmin Pohoata, Tudor Vianu National College of Informatics, Bucharest, Romania.*

Let M be a point on the circumcircle of triangle ABC and lying on the arc BC that does not contain A. Let I be the incenter of ABC, and let E and F be the feet of the perpendiculars from I to lines MB and MC, respectively. Prove that the value of

$$\frac{IE+IF}{AM}$$

is independent of the position of M.

I. *Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

Let $\theta = \angle CBM$. Then, because triangles BEI and IFC are right triangles, we have

$$\frac{IE}{IB} = \sin\left(\frac{B}{2}+\theta\right) \quad\text{and}\quad \frac{IF}{IC} = \sin\left(\frac{C}{2}+A-\theta\right) = \cos\left(\frac{B}{2}-\frac{A}{2}+\theta\right).$$

In addition, if r denotes the inradius of triangle ABC, then

$$r = IB\sin\left(\frac{B}{2}\right) \quad\text{and}\quad r = IC\sin\left(\frac{C}{2}\right).$$

Therefore

$$\begin{aligned}&\frac{IE+IF}{r}\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\\ &\quad= \sin\left(\frac{C}{2}\right)\sin\left(\frac{B}{2}+\theta\right)+\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}-\frac{A}{2}+\theta\right)\end{aligned}$$

$$= \cos\left(\frac{A}{2}+\frac{B}{2}\right)\sin\left(\frac{B}{2}+\theta\right)+\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}-\frac{A}{2}+\theta\right)$$
$$= \frac{1}{2}\left(\sin\left(B+\theta+\frac{A}{2}\right)-\sin\left(\frac{A}{2}-\theta\right)+\sin\left(B+\theta-\frac{A}{2}\right)+\sin\left(\frac{A}{2}-\theta\right)\right)$$
$$= \sin(B+\theta)\cos\left(\frac{A}{2}\right).$$

By the Law of Sines applied to triangle AMB, we have $AM = 2R\sin(B+\theta)$, where R is the circumradius of triangle ABC. It follows that

$$\frac{IE+IF}{AM} = \frac{r\cos(A/2)}{2R\sin(B/2)\sin(C/2)},$$

which is independent of θ and thus independent of the position of M. This concludes the proof, but further simplification is possible. Indeed, because

$$\frac{r}{4R} = \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right),$$

we find that

$$\frac{IE+IF}{AM} = \sin A.$$

II. *Solution by Victor Kutsenok, University of St. Francis, Fort Wayne, IN.*

Let T and V be the feet of the perpendiculars from A to lines MB and MC, respectively. See FIGURE 1, below. Let I be the incenter of triangle ABC and let K and L be the feet of the perpendiculars from I to lines AT and AV, respectively. Then $KT = IE$ and $LV = IF$. Because quadrilaterals $ATMV$ and $ABMC$ are cyclic, we have $\angle TAV = \pi - \angle TMC = \angle BAC$. It follows that $\angle BAT = \angle CAV$, so right triangles ATB and AVC are similar. If follows that triangles ABC and ATV are similar with sides in the ratio of

$$\frac{AT}{AB} = \frac{AV}{AC} = \cos CAV.$$

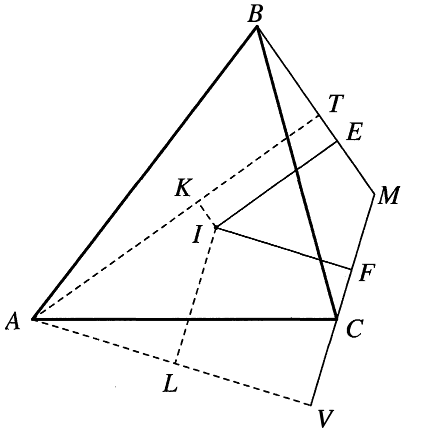

Figure 1

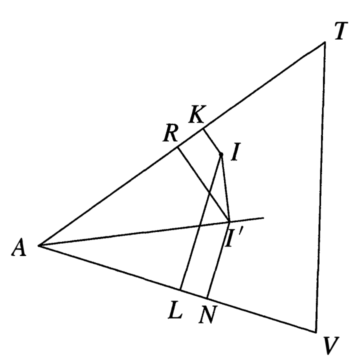

Figure 2

Now consider the similarity transformation $\mathcal{T}$ that maps triangle ABC onto triangle ATV. This transformation can be described as a rotation about A through $\angle CAV$, then a dilation with magnification factor $\cos CAV$ and centered at A. Note that if $P \neq A$ and $\mathcal{T}(P) = P'$, then

$$\frac{AP'}{AP} = \cos CAV \qquad \text{and} \qquad \angle P'AP = \angle VAC.$$

It then follows that $\angle AP'P$ is a right angle.

Now let $\mathcal{T}(I) = I'$, let N and R be the feet of the perpendiculars from I' to lines AV and AT, respectively. See FIGURE 2. Note that I' is the incenter of triangle TAV. Because II' is perpendicular to AI', and AI' is the bisector of $\angle TAV$, it follows that II' has congruent orthogonal projections onto lines AV and AT, that is, $KR = LN$. Because I' is the incenter of triangle TAV and I is the incenter of triangle ABC, we have

$$TV = RT + NV = (KT \pm KR) + (LV \mp LN) = KT + LV = IE + IF.$$

Therefore

$$\frac{IE + IF}{AM} = \frac{TV}{AM} = \frac{AM \sin BAC}{AM} = \sin BAC,$$

which does not depend on the position of M.

Also solved by George Apostolopoulos (Greece), Michel Bataille (France), Robert Calcaterra, Miguel Amengual Covas (Spain), Chip Curtis, Dmitry Fleischman, Michael Goldenberg and Mark Kaplan, G.R.A.20 Problem Solving Group (Italy), Yoo Kyung Jeon (Korea), Victor Y. Kutsenok, Kee-Wai Lau (China), Robin Oakapple, Ossama A. Saleh and Terry J. Walters, Joel Schlosberg, John Sumner and Aida Kadic-Galeb, Michael Vowe (Switzerland), and the proposer.

A division ring **December 2008**

1810. *Proposed by Greg Oman, Otterbein College, Westerville, OH.*

Let R be a ring. For elements $x, y \in R$, we say x divides y on the right if and only if there is a $z \in R$ with $xz = y$. (We denote this by $x|_r y$.) An element $p \in R$ is a right prime if and only if whenever $p|_r xy$, then either $p|_r x$ or $p|_r y$. Prove that if every element of R is right prime, then R is a division ring, that is, the nonzero elements of R form a group under multiplication. (Note: R is not assumed to be commutative nor is it assumed that R has a multiplicative identity.)

Solution by Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.

First, suppose x and y are elements of R such that $xy = 0$. Then $0 \cdot 0 = xy$ and since 0 is right prime, then either $0|_r x$ or $0|_r y$. Thus either $x = 0$ or $y = 0$ and it follows that the left and right cancellation laws hold in R. For the remainder of the proof, assume that x is a nonzero element of R. Because $xx = xx$, we have $x|_r x$, so there is an element e such that $xe = x$. Clearly, $0e = e0 = 0$. Suppose y is any nonzero element of R. Then $xey = xy$ and hence $ey = y$. In particular, $ex = x$. It follows that $yx = yex$ and thus $y = ye$. Therefore, e is the multiplicative identity of R. Since $x^2e = xx$, we see that $x^2|_r x$. Thus there is an element z such that $x^2z = x$. This implies that $xz = e$. But then $xzx = x = x^2z$, which implies that $zx = xz = e$. Thus every nonzero element of R has a multiplicative inverse, so R is a division ring.

Also solved by Armstrong Problem Solvers, Paul Budney, Robert Calcaterra, Toni Ernvall (Finland), John Ferdinands, J. Fredette and M. Winders, Lee O. Hagglund, Stephen J. Herschkorn, Elias Lampakis (Greece), David P. Lang, Auturo Magidin, Missouri State University Problem Solving Group, Northwestern University Math Problem Solving Group, Paul Peck, John Sumner and Aida Kadic-Galeb, Tony Tam, Marian Tetiva (Romania), Bob Tomper, Ken Yanosko, and the proposer.

Editor's Note. A few readers pointed out that the solution to Problem 1783 that appeared in the December 2008 issue was incorrect. A correct solution appears below. Thanks to observant readers for pointing out the error.

An inequality of reciprocals **December 2007**

1783. *Proposed by Ovidiu Bagasar, Babes Bolyai University, Cluj Napoca, Romania.*

Let n be a positive integer and let $x_1, x_2, \dots, x_n$ be positive real numbers. Let $S = x_1^n + x_2^n + \cdots + x_n^n$ and $P = x_1 x_2 \cdots x_n$. Prove that

$$\sum_{k=1}^{n} \frac{1}{S - a_k^n + P} \leq \frac{1}{P}.$$

Solution by Lucyna Kabza, Southeastern Louisiana University, Hammond, LA.

We prove that

$$\frac{1}{P} - \sum_{k=1}^{n} \frac{1}{S - x_k^n + P} \geq 0.$$

We have

$$\begin{aligned} \frac{1}{P} - \sum_{k=1}^{n} \frac{1}{S - x_k^n + P} &= \sum_{k=1}^{n} \left(\frac{1}{nP} - \frac{1}{S - x_k^n + P} \right) \\ &= \sum_{k=1}^{n} \frac{S - x_k^n + P - nP}{nP\left(S - x_k^n + P\right)} \\ &\geq \sum_{k=1}^{n} \frac{S - x_k^n + P - nP}{nP\,(S + P)} \\ &= \frac{1}{nP(S+P)} \left((n-1)S - n(n-1)P\right) \\ &= \frac{n-1}{nP(S+P)} (S - nP). \end{aligned}$$

Now observe that

$$\frac{S}{n} = \frac{x_1^n + x_2^n + \cdots + x_n^n}{n} \geq \sqrt[n]{P^n} = P, \quad \text{or equivalently} \quad S \geq nP.$$

This completes the proof.

Answers

Solutions to the Quickies from page 383.

A995. We use the method of "mass points." Place point masses of mass λ at each of B and C and one of mass 1 at A. Let P be the position of the center of mass of this 3 mass system. We calculate the position of P in two ways. First note the system of masses at B and C has center of mass at D and this system is equivalent to a mass of 2λ at the point D. It follows that P is on AD. Next note that the system of masses at A and B has center of mass at E and is equivalent to a mass of $1 + \lambda$ at E. It follows that

P lies on CE. Therefore, $P = F$. Next observe that F (or P) is the center of mass of the two point system consisting of a mass of $1+\lambda$ at E and a mass of 1 at C. By the Law of Moments, $(1+\lambda)FE = \lambda \cdot CF$. Therefore

$$\frac{1+\lambda}{\lambda} = \frac{CF}{FE} = \lambda,$$

and it follows that $\lambda = \frac{1+\sqrt{5}}{2}$, the golden mean.

A996.

a. The answer is no, as demonstrated by $f(z) = \bar{z}$.

b. The answer is yes. First note that because $f(0) = f(0+0) = f(0) + f(0)$, we must have $f(0) = 0$. Next let z any complex number. Then $f'(z)$ exists because

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z) + f(\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = f'(0).$$

Therefore $f'(z) \equiv a$, where $a = f'(0)$. If follows that $f(z) = ax + b$ for some constant b. The condition $f(0) = 0$ implies that $b = 0$.

To appear in *The College Mathematics Journal*, January 2010

Articles

Under-represented Then Over-represented: A Memoir of Jews in American Mathematics, *by Reuben Hersh*

Dogs Don't Need Calculus, *by Michael Bolt and Daniel C. Isaksen*

The Helen of Geometry, *by John Martin*

Biangular Coordinates Redux: Discovering a New Kind of Geometry, *by Michael Naylor and Brian Winkel*

An Upper Bound for the Expected Range of a Random Sample, *by Manuel Lopez and James Marengo*

The Hardest Straight-In Pool Shot, *by Rick Mabry*

Classroom Capsules

Computing Definite Integrals using the Definition *by Jim Hartman*

Waiting to Turn Left? *by Maureen T. Carroll, Elyn K. Rykken, and Jody M. Sorensen*

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

doi:10.4169/002557009X478454

Gardner, Martin, *The Jinn from Hyperspace: And Other Scribblings—Both Serious and Whimsical*, Prometheus Books, 2008; 307 pp, $25.98. ISBN 978-1-59102565-8. Pegg, Ed, Jr., Alan H. Schoen, and Tom Rodgers (eds.), *Homage to a Pied Puzzler*, and *Mathematical Wizardry for a Gardner*, A K Peters, 2009; each xix + 285 pp, $49. ISBN 978-1-56881-315-8, 978-1-56881-447-6.

Martin Gardner, noted philosopher and for 25 years author of the column "Mathematical Games" in *Scientific American*, has turned 95—and has turned out just about that many books, many of them about mathematics. This collection of his essays is divided into four parts: Science, Math, and Baloney (19 essays), Literature (5, including 2 on Chesterton), L. Frank Baum (7 on the author of *The Wizard of Oz*), and Lewis Carroll (5). The mix of subjects is entertaining, enlightening, and educational. The mathematical topics include the Banach-Tarski paradox, a genie in a Klein bottle, transcendental numbers, several kinds of paradoxes, and a notable essay in defense of mathematical platonism. A few essays are new, but most are reprinted, from diverse sources. The essay on platonic realism appeared as a book review in *Notices of the American Mathematical Society* 52 (11) (December 2005) 1344–1347 (not in the September issue as mentioned by Gardner); the book includes an addendum and a further postscript. The two volumes from Pegg et al. contain papers on all kinds of recreational mathematics and magic by participants at the 2006 biennial conference held in honor of Gardner. (Puzzle from this book: Who is the author "Earnest Hammingway" of the piece "Horses in the stream ... "? A Google search turns up only misspelled references to the Nobelist novelist.)

Butler, Steve, Mohammad T. Hajiaghay, Robert D. Kleinberg, and Tom Leighton, Hat guessing games, *SIAM Review* 51 (2) (2009) 399–413. Hardin, Christopher, and Alan D. Taylor, An introduction to infinite hat problems, *Mathematical Intelligencer* 30 (2008) (4) 20–25.

Among the puzzles popularized by Martin Gardner is a classic about colored hats:

> "Three men—A, B and C—are blindfolded and told that either a red or a green hat will be placed on each of them. After this is done, the blindfolds are removed; the men are asked to raise a hand if they see a red hat, and to leave the room as soon as they are sure of the color of their own hat. All three hats happen to be red, so all three men raise a hand. Several minutes go by until C, who is more astute than the others, leaves the room. How did he deduce the color of his hat?"
>
> —*The 2nd Scientific American Book of Mathematical Puzzles & Diversions* (1961) 123, originally in *Scientific American* (February 1959) 136.

Well, as the cited articles manifest, 50 years later this particular vein of recreational mathematics is scarcely mined out! Moreover, hat problems have connections to Hamming codes and to design of auctions, as well as in computer science (communication complexity). Butler et al. begin with an optimal strategy to the classic problem for n players and hats of k colors, then develop "guessing graphs" and "sight graphs." Hardin and Taylor extend Butler et al.'s results for the finite case before letting the Axiom of Choice help deal with infinite ones.

Hill, Theodore P., Knowing when to stop: How to gamble if you must—the mathematics of optimal stopping, *Scientific American* 97 (2) (March-April 2009) 126–133.

Author Hill begins with what used to be known as the secretary problem, but also may apply to hiring a new member in your department: Interview successive candidates, whom you either hire (and stop looking) or else lose; what strategy maximizes the probability of hiring the best? Hill reveals a way to raise the probability above 50% when there are only two candidates. (But will your dean go along with a solution that s/he doesn't understand?) Hill considers also problems with complete information, treating them via backward induction (otherwise known as dynamic programming), and concludes with a simply-stated but so far unsolved problem.

Fulda, Joseph S., Perfectly marked, fair tests with unfair marks, *Mathematical Gazette* 93 (No. 527) (July 2009) 256–260.

For different fair tests—all on the same material, each area covered proportionately, a level in accord with the teaching, graded perfectly—a student's score would differ from one test to another. Moreover, different students would have different orderings. That is, tests differ from student to student in accuracy of assessment, which seems unarguable. Fulda asserts that in giving such a fair test, a teacher nevertheless is unfair (intentionally, subconsciously, or unintentionally). An ethical teacher may not give inaccurate grades and hence "must adjust ... grades he knows are inaccurately low." However, to avoid the perception of injustice, adjustments must be only upwards. (On occasion I have done that.) Still, even then, there will be unfair grades—just not unfairly low ones. The only way out of the subjectivity of grading, he claims, is for the examiner not to be the teacher. This "solution" seems to me not to address inaccurate grades; it just passes the ethical buck, away from the teacher, to advocates of standardized testing.

Szydlik, Stephen D., The problem with the snack food problem: Solving systems of linear equations when using real data can present surprising challenges, *Mathematics Teacher* 103 (1) (August 2009) 18–25.

Author Szydlik relates an exercise in linear algebra (which I use to illustrate multiple regression). Students choose different snack foods; record the number of grams of fat, carbohydrates, protein, and calories in a standard serving; and are asked to determine the number of calories per gram of each of the three categories. Three unknowns are usually uniquely determined by three conditions—the data from three snack foods. However, student solutions can vary greatly from the accepted caloric values for fat, carbohydrates, and protein (9, 4, 4): For instance, how about $(8.7, 5.1, -10.0)$? Szydlik leads students to examine the stability of the system of equations and illustrates the situation with graphs of the planes represented by the equations. Moreover, he provides an explanation for the instability in terms of the nutritional aspects of the snack foods chosen: They are all high-fat, high-carb, and low-protein. Replacing one with a high-protein snack produces a more stable mathematical system—and a more balanced diet.

Dekking, Michel, and André Hensbergen, A problem with the assessment of an iris identification system, *SIAM Review* 51 (2) (2009) 417–422.

In the future, you may be identified by the iris of your eye. The authors investigate a system devised by John Daugman (Cambridge University) that uses wavelet coefficients to code an iris scan into a bit sequence, or codeword, of 2,048 bits. However, two scans of the same eye do not necessarily produce identical codewords. A natural measure of the difference between two codewords is the Hamming distance, the number of bits that disagree. Daugman modeled disagreement at each bit by an independent Bernoulli random variable, giving the Hamming distance a binomial distribution. Daugman applied data from 2,000 scans, taken in all $\binom{2000}{2} \approx 2$ million pairs, to calculate a dividing point between false acceptance and false rejection, each with probability one in a million. Dekking and Hensbergen critique Daugman's analysis (e.g., the pairs are not independent) and provide their own calculation of the expected variance of a Hamming distance for such a pair. Their work, however, eventually supports Daugman's claims, after pleasant calls on Lagrange's identity, the Chebyshev order inequality, and the Cauchy-Schwarz inequality. (Dekking and Hensbergen's use of *code* instead of *codeword* is confusing; but did you know that *iris* has as plurals both *irises* and *irides*? I didn't.)

NEWS AND LETTERS

Acknowledgments

In addition to our Associate Editors, the following referees have assisted the MAGAZINE *during the past year. We thank them for their time and care.*

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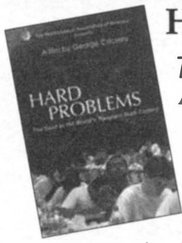
PROBLEMS

The letters P, Q, and S refer to Proposals, Quickies, and Solutions, respectively; page numbers appear in parentheses. For example, P1817 (147) refers to Proposal 1817, which appears on page 147.

Thanks to Problems Editor Elgin Johnston

With this issue, Elgin Johnston completes nine years of service as Problems Editor. Please take a moment to ponder the creative energy that this volunteer professional brought to our pages: He offered readers a string of challenging problems, from number 1613 in his first column to 1835 in this issue. He sorted and carefully edited the best solutions to problems 1589 through 1810. And let us not forget the entertaining string of Quickies. Our community owes Elgin Johnston hearty congratulations and sincere thanks.



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1. Publication Title: Mathematics Magazine
2. Publication Number: 0025-570X
3. Filing Date: 9/4/09
4. Issue Frequency: Bimonthly Except July+August
5. Number of Issues Published Annually: 5
6. Annual Subscription Price: $49
7. Complete Mailing Address of Known Office of Publication (Not printer) (Street, city, county, state, and ZIP+4®): Mathematical Assn of America, 1529 18th St, N.W., Washington, D.C. 20036
Contact Person: H. Waldman
Telephone (Include area code): 202-387-5200
8. Complete Mailing Address of Headquarters or General Business Office of Publisher (Not printer): Mathematical Assn of America, 1529 18th St. N.W., Washington, D.C. 20036
9. Full Names and Complete Mailing Addresses of Publisher, Editor, and Managing Editor (Do not leave blank)
Publisher (Name and complete mailing address): Mathematical Assn of America, 1529 18th St, N.W., Washington, D.C. 20036
Editor (Name and complete mailing address): Frank Farris, Santa Clara University, Santa Clara, CA 95053
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13. Publication Title: Mathematics Magazine
14. Issue Date for Circulation Data Below: June 2009

15. Extent and Nature of Circulation		Average No. Copies Each Issue During Preceding 12 Months	No. Copies of Single Issue Published Nearest to Filing Date
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i. Percent Paid (15c divided by 15f times 100)		99%	99%

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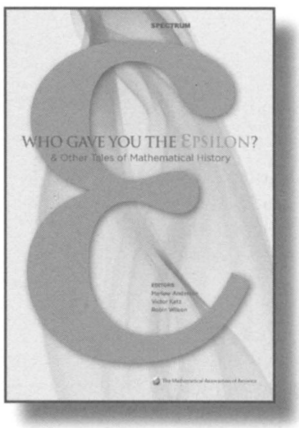

Marlow Anderson, Victor Katz, & Robin Wilson, Editors

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